

Signals and Systems

Fall 2003

Lecture #13

21 October 2003

1. The Concept and Representation of Periodic Sampling of a CT Signal
2. Analysis of Sampling in the Frequency Domain
3. The Sampling Theorem — the Nyquist Rate
4. In the Time Domain: Interpolation
5. Undersampling and Aliasing

SAMPLING

We live in a continuous-time world: most of the signals we encounter are CT signals, e.g. $x(t)$. How do we convert them into DT signals $x[n]$?

— Sampling, taking snap shots of $x(t)$ every T seconds.

T – sampling period

$x[n] \equiv x(nT)$, $n = \dots, -1, 0, 1, 2, \dots$ — regularly spaced samples

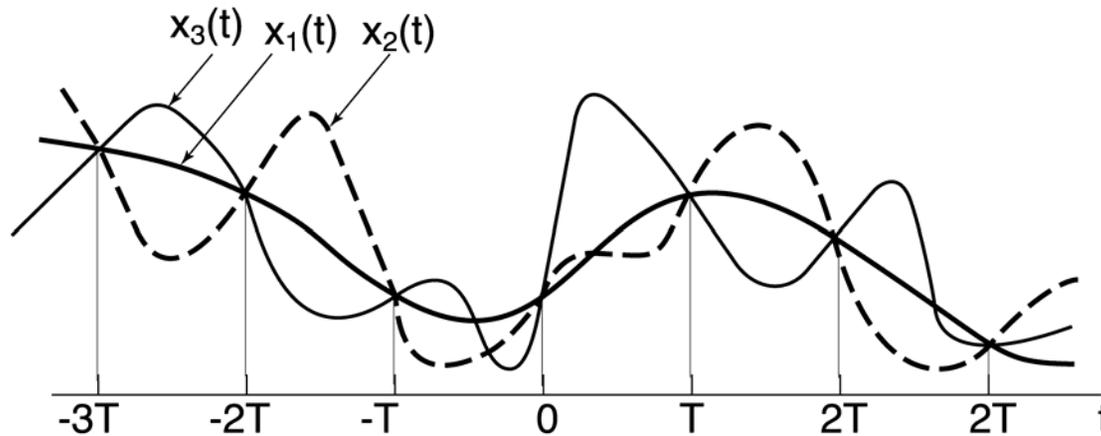
Applications and Examples

- Digital Processing of Signals
- Strobe
- Images in Newspapers
- Sampling Oscilloscope
- ...

How do we perform sampling?

Why/When Would a Set of Samples Be Adequate?

- Observation: *Lots* of signals have the same samples

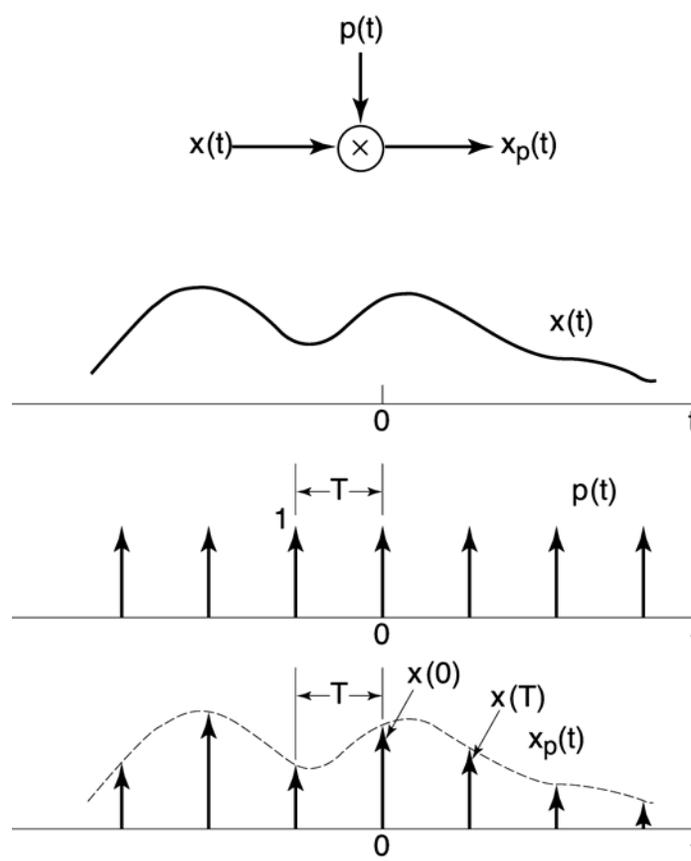


- By sampling we throw out lots of information
 - all values of $x(t)$ between sampling points are lost.
- **Key Question for Sampling:**
Under what conditions can we **reconstruct** the original CT signal $x(t)$ from its samples?

Impulse Sampling — Multiplying $x(t)$ by the sampling function

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = x(t)p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$



Analysis of Sampling in the Frequency Domain

$$x_p(t) = x(t) \cdot p(t)$$

$$\text{Multiplication Property} \Rightarrow X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

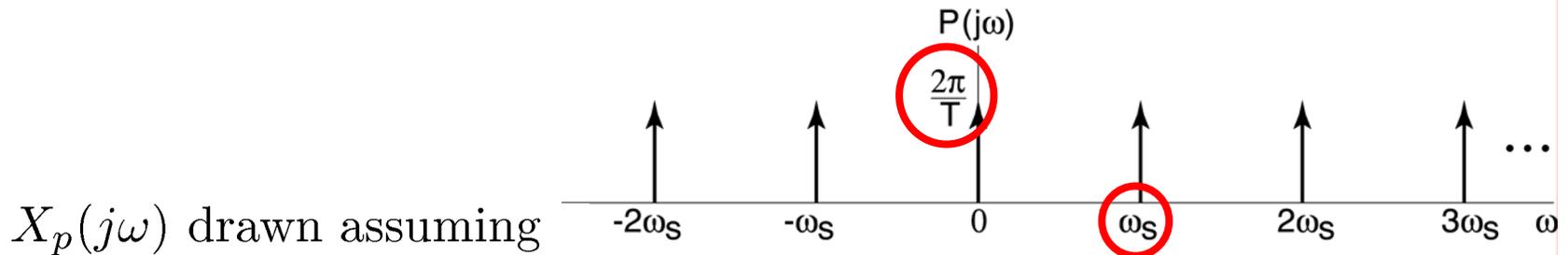
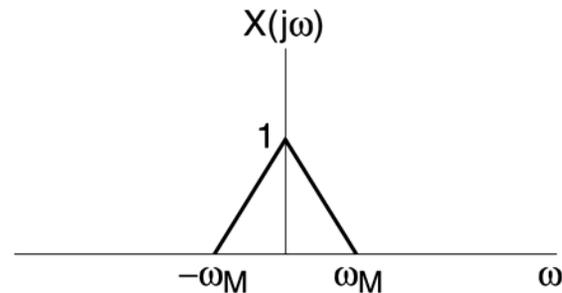
$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$\omega_s = \frac{2\pi}{T} = \text{Sampling Frequency}$$

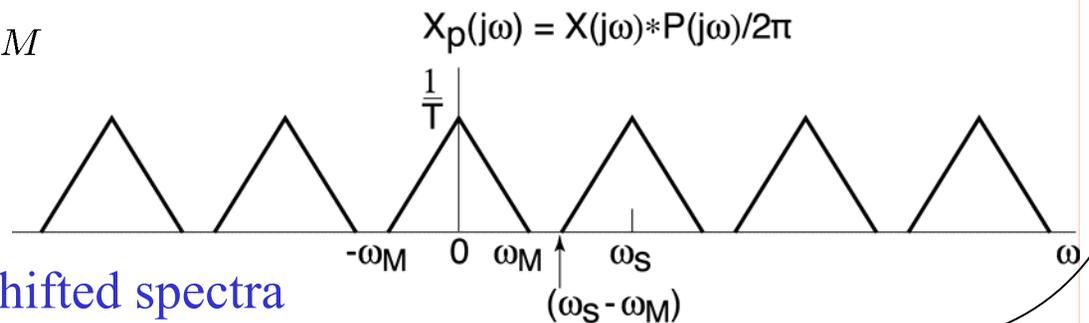
Important to
note: $\omega_s \propto 1/T$

$$\begin{aligned} X_p(j\omega) &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j\omega) * \delta(\omega - k\omega_s) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \end{aligned}$$

Illustration of sampling in the frequency-domain for a band-limited ($X(j\omega)=0$ for $|\omega|> \omega_M$) signal

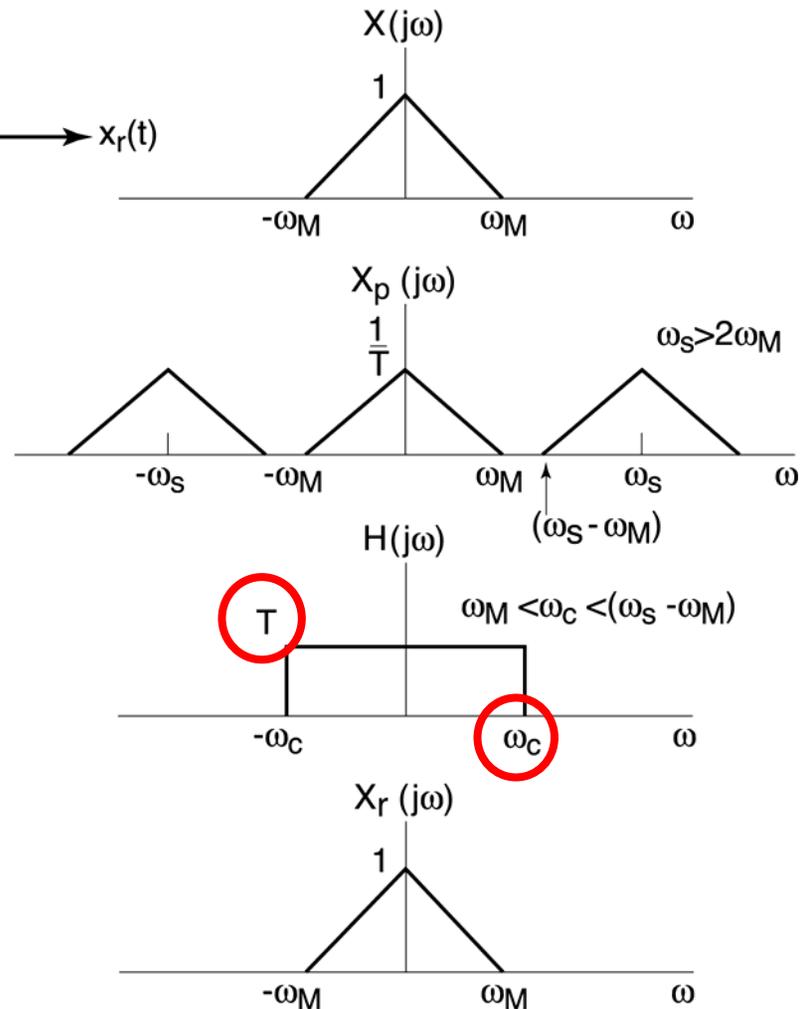
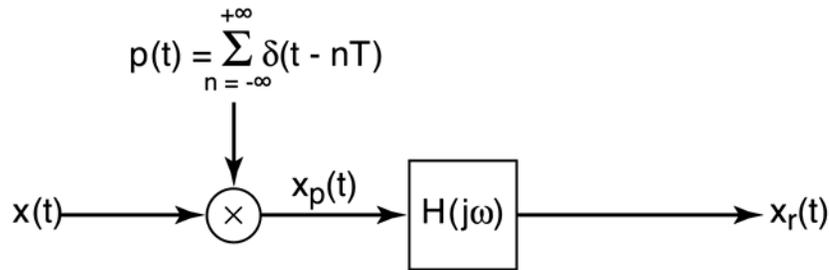


$\omega_s - \omega_M > \omega_M$
i.e. $\omega_s > 2\omega_M$



No overlap between shifted spectra

Reconstruction of $x(t)$ from sampled signals



If there is no overlap between shifted spectra, a LPF can reproduce $x(t)$ from $x_p(t)$

The Sampling Theorem

Suppose $x(t)$ is bandlimited, so that

$$X(j\omega) = 0 \quad \text{for } |\omega| > \omega_M$$

Then $x(t)$ is uniquely determined by its samples $\{x(nT)\}$ if

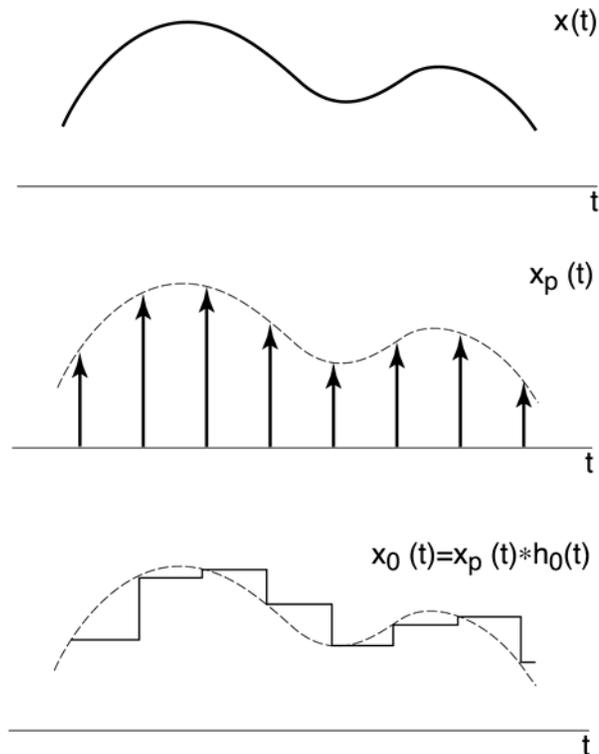
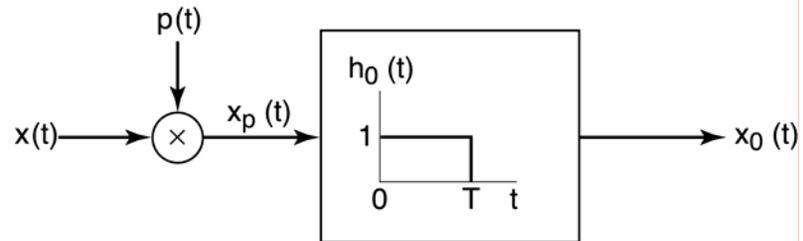
$$\omega_s > 2\omega_M = \text{The Nyquist rate}$$

$$\text{where } \omega_s = 2\pi/T$$

Observations on Sampling

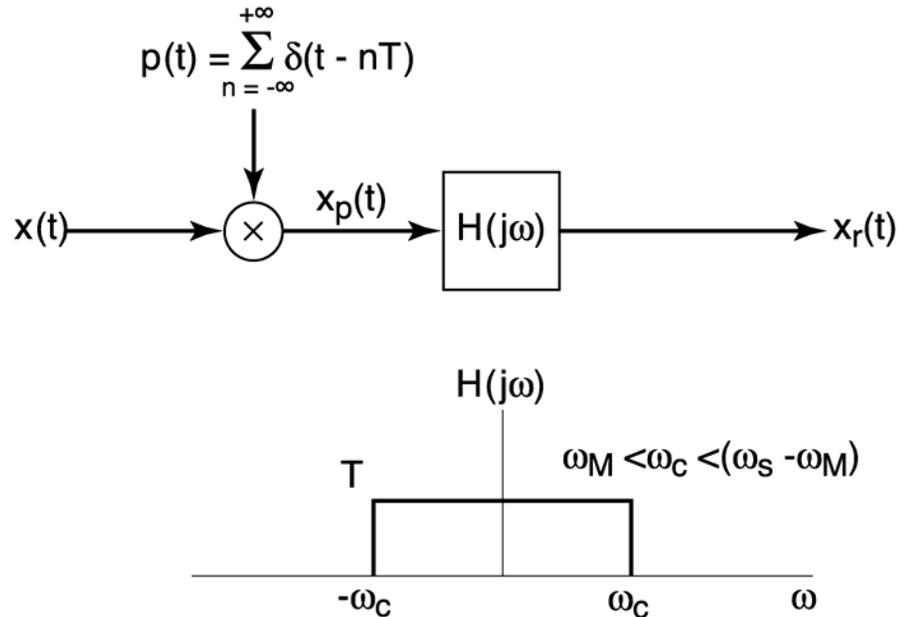
(1) In practice, we obviously don't sample with impulses or implement ideal lowpass filters.

— One practical example:
The Zero-Order Hold



Observations (Continued)

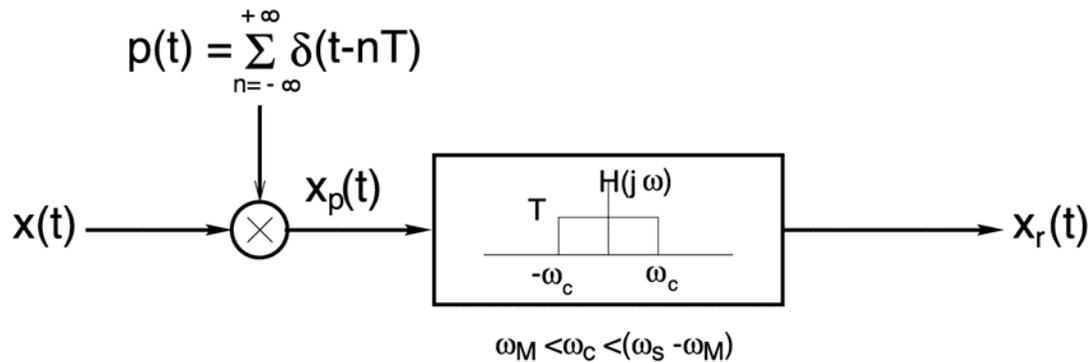
- (2) Sampling is fundamentally a *time-varying* operation, since we multiply $x(t)$ with a time-varying function $p(t)$. However,



is the identity system (which is *TI*) for bandlimited $x(t)$ satisfying the sampling theorem ($\omega_s > 2\omega_M$).

- (3) What if $\omega_s \leq 2\omega_M$? Something different: more later.

Time-Domain Interpretation of Reconstruction of Sampled Signals — Band-Limited Interpolation

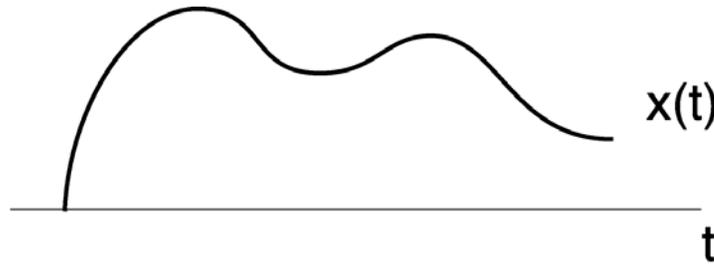


$$\begin{aligned}
 x_r(t) &= x_p(t) * h(t) \quad , \quad \text{where } h(t) = \frac{T \sin \omega_c t}{\pi t} \\
 &= \left(\sum_{n=-\infty}^{\infty} x(nT) \delta(t - nT) \right) * h(t) \\
 &= \sum_{n=-\infty}^{\infty} x(nT) h(t - nT) = \sum_{n=-\infty}^{\infty} x(nT) \frac{T \sin[\omega_c(t - nT)]}{\pi(t - nT)}
 \end{aligned}$$

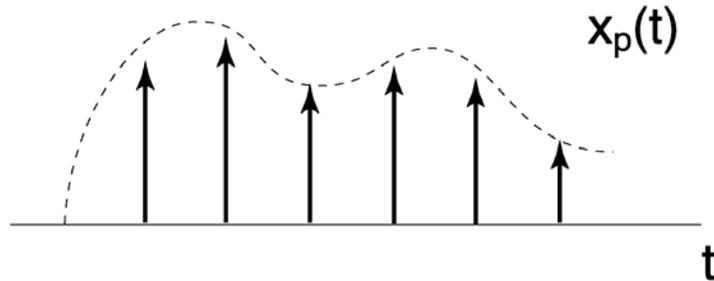
The lowpass filter interpolates the samples *assuming* $x(t)$ contains no energy at frequencies $\geq \omega_c$

Graphic Illustration of Time-Domain Interpolation

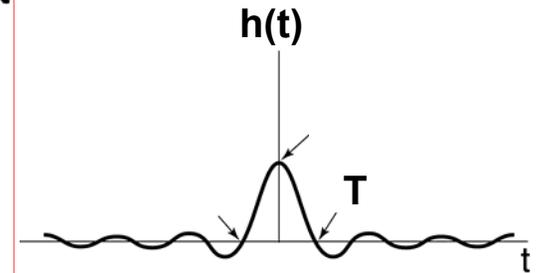
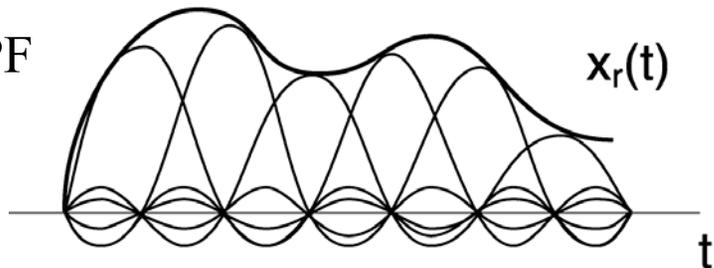
Original
CT signal



After sampling

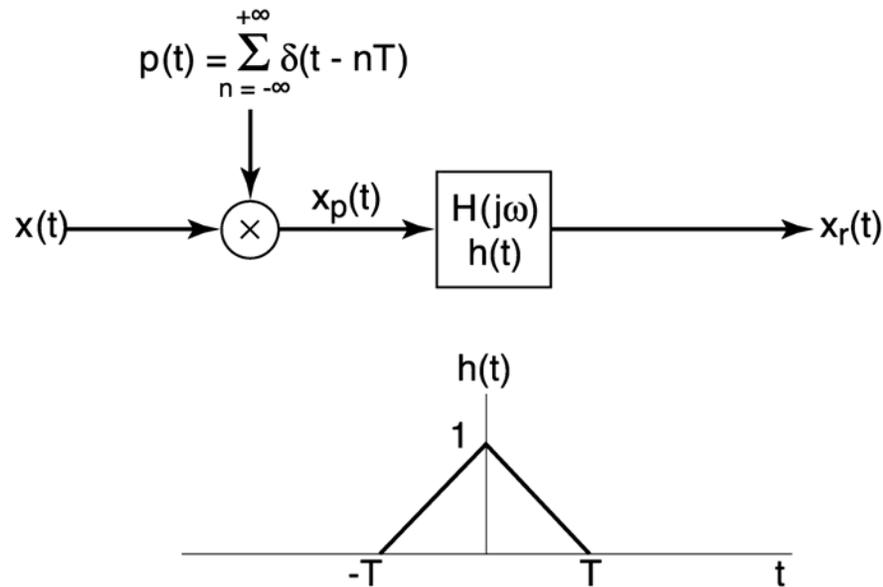
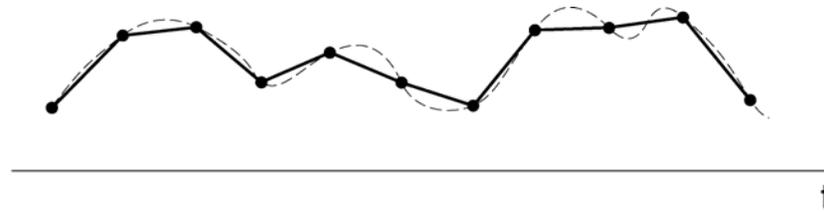


After passing the LPF



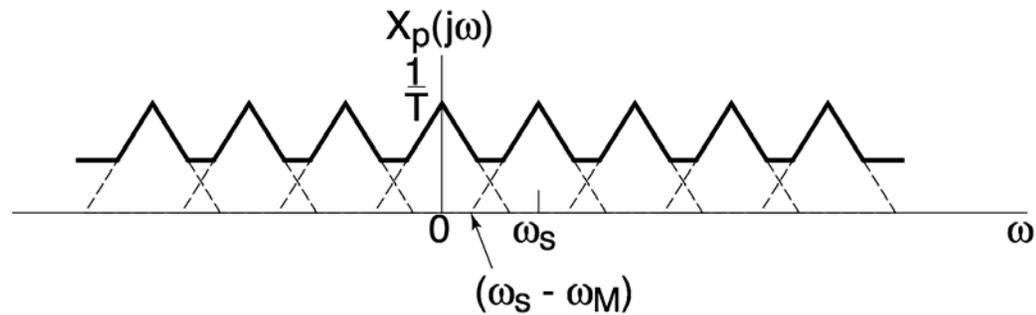
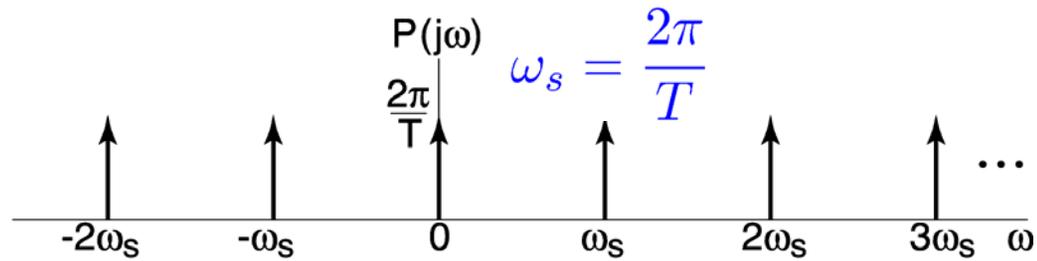
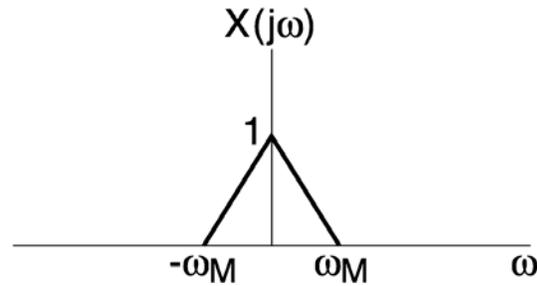
Interpolation Methods

- Bandlimited Interpolation
- Zero-Order Hold
- First-Order Hold — Linear interpolation

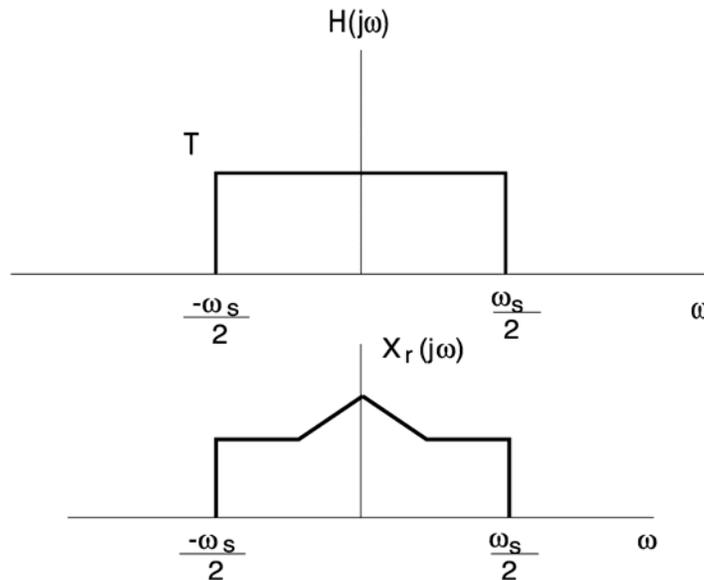
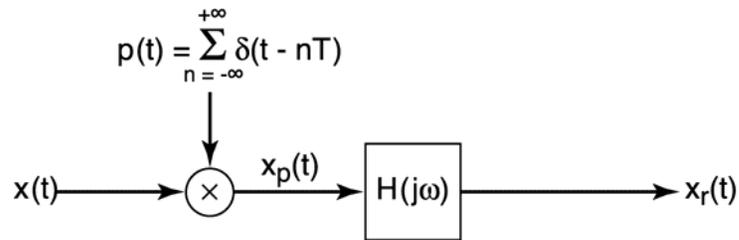


Undersampling and Aliasing

When $\omega_s \leq 2 \omega_M \Rightarrow$ Undersampling



Undersampling and Aliasing (continued)

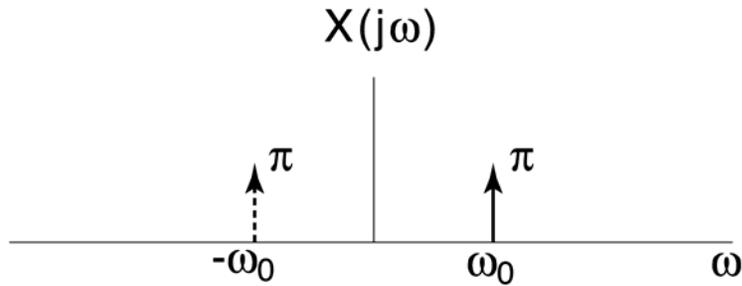


$X_r(j\omega) \neq X(j\omega)$
Distortion because
of *aliasing*

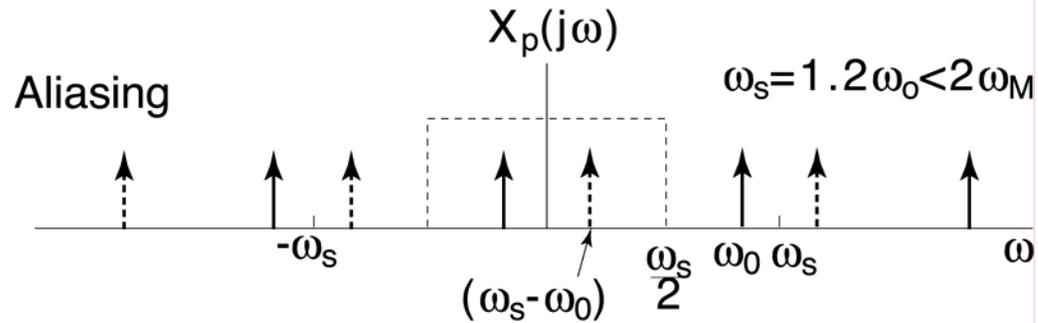
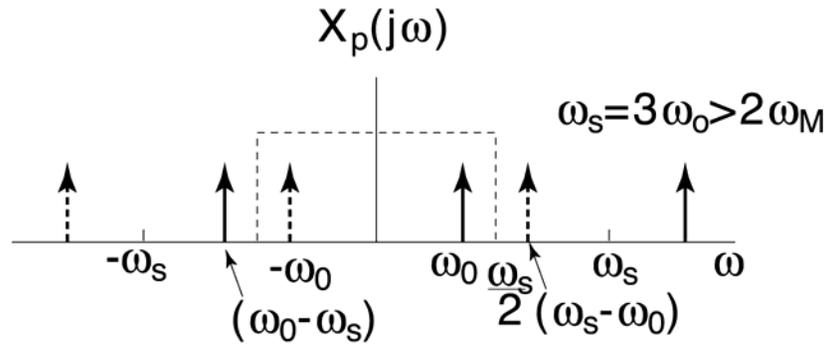
- Higher frequencies of $x(t)$ are “folded back” and take on the “aliases” of lower frequencies
- Note that at the sample times, $x_r(nT) = x(nT)$

A Simple Example

$$x(t) = \cos(\omega_0 t + \phi)$$



Picture would be Modified...



Demo: Sampling and reconstruction of $\cos \omega_0 t$