Lecture 3

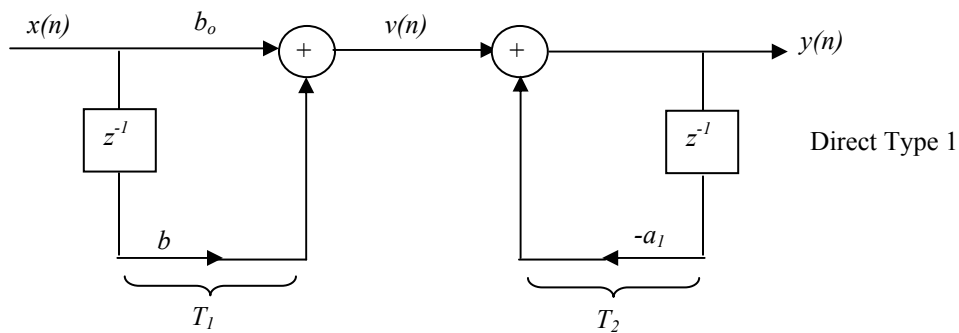### Implementation of Discrete-Time Systems
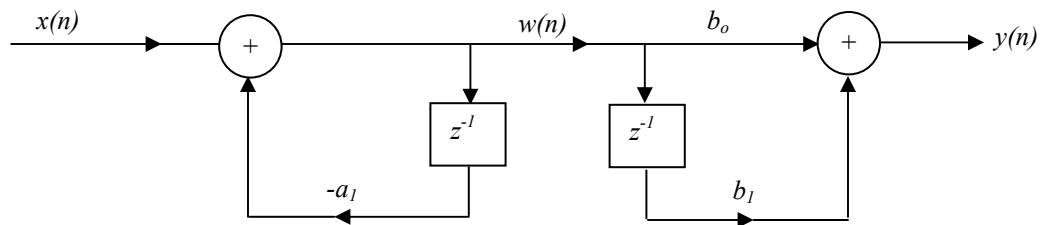
Lets start with 1st order system:

$$y(n) = -a_1 y(n-1) + b_o x(n) + b_1 x(n-1) \text{ or } \sum_{k=0}^{M} a_k y(n-k) = \sum_{k=0}^{N} b_k x(n-k)$$

This can be considered as $v(n) = b_o x(n) + b_1 x(n-1)$ and

$$y(n) = a_1 y(n-1) + v(n)$$
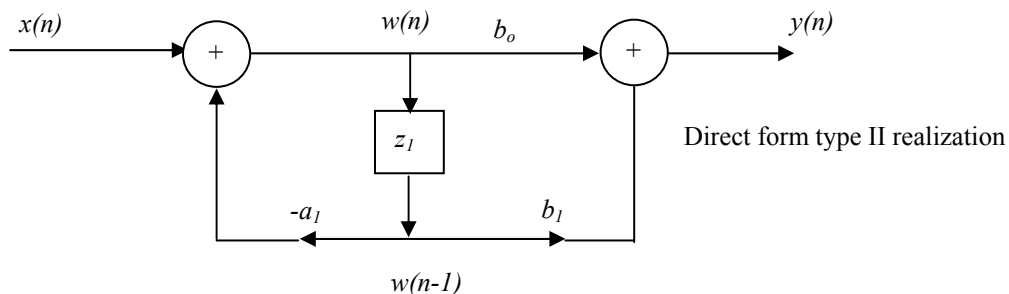

Direct Type 1

But for the LTI system, we can interchange the order and write $T = T_2 \bullet T_1 = T_1 \bullet T_2$

From this figure we can write: $w(n) = -a_1 w(n-1) + x(n)$

$$y(n) = b_o w(n) + b_1 w(n-1)$$

Now we can combine the two delays and get

Direct form type II realization

So in general form $\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{N} b_k x(n-k)$ if $N > M$ (always $a_o = 1$) can be represented as the following.

Correlation of Discrete-Time Signals

Cross-Correlation

Definition:

$$r_{xy}(\ell) = \sum_{n=\infty}^{+\infty} x(n)y(n-\ell) \qquad \ell = 0, \pm 1, \pm 2$$

$$= \sum_{n=\infty}^{+\infty} x(n+\ell)y(n)$$

$$= x(\ell) * y(-\ell)$$

$r_{yx}(\ell) = r_{xy}(-\ell)$

When $y(n) = x(n)$, then it is called auto-correlation.

A symmetric (even) function $r_{xx}(\ell) = \sum_{n=\infty}^{+\infty} x(n)x(n-\ell) = \sum_{n=\infty}^{+\infty} x(n)x(n+\ell) = x(n) * x(-n)$

Properties

$$r_{xx}(0) = \sum_{ns-\infty}^{+\infty} x(n) \bullet x(n) = \sum_{ns-\infty}^{+\infty} |x(n)|^2 = E_x \quad \text{energy}$$

$$|r_{xy}(\ell)| \leq \sqrt{r_{xx}(0)r_{yy}(0)} = \sqrt{E_x E_y}$$

and
$$\left|r_{xx}(\ell)\right| \le r_{xx}(o) = E_x$$

Normalized auto or cross correlation:

$$\rho_{xx}(\ell) = \frac{\gamma_{xx}(\ell)}{r_{xx}(0)} \le 1 \qquad \rho_{xy}(\ell) = \frac{r_{xy}(\ell)}{\sqrt{r_{xx}(0)r_{yy}(0)}} \le 1$$

For periodic signals, the correlation function is defined in one period:

$$r_{xy}(\ell) = \lim_{M \to \infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x(n)y(n-\ell),$$

where M is the number of observed samples.

If x and y are both periodic with period N, then

$$r_{xy}(\ell) = \frac{1}{N} \sum_{n=o}^{N-1} x(n)y(n-\ell) \Rightarrow \text{correlation is also a periodic sequence with period } N.$$

Question: Consider a signal $x(n) = \cos 2\pi\left(\frac{1}{3}\right)n$ for 100 samples. Its autocorrelation is periodic with the same $\frac{1}{3}$ frequency but its magnitude decreases as ℓ increases. Why?

Answer: Because we have a finite set of data recorded at M samples. So many of the products of $x(n)x(n-\ell)$ are zero when ℓ increases. Therefore, we should avoid computing $r_{xx}|\ell|$ for large lags, say $\ell > \frac{M}{2}$.

A Practical Application at Usage of Autocorrelation Function

Lets say we have a signal with unknown period N and it is corrupted by additive white noise. Then $r_{xx}(\ell)$ can be used to detect its periodicity $y(n) = x(n) + w(n)$.

$$r_{yy}(\ell) = \frac{1}{M} \sum_{n=o}^{M-1} y(n)y(n-\ell) \qquad M \gg N$$

$$= \frac{1}{M} \sum_{n=o}^{M-1} \left[x(n) + w(n)\right]\left[x(n-\ell) + w(n-\ell)\right]$$

$$= \frac{1}{M} \sum_{n>o}^{M-1} \left[x(n)x(n-\ell) + x(n)w(n-\ell) + w(n)x(n-\ell) + w(n)w(n-\ell)\right]$$

$$= r_{xx}(\ell) + r_{ww}(\ell) + r_{xw}(\ell) + r_{wx}(\ell)$$

If $w(n)$ is a white noise then it has value only at lag 0 (only $r_{ww}(0)$ is not zero). r_{xw} and r_{wx} are almost zero and therefore, only $r_{xx}(\ell)$ is showing some peaks for every period.

Solving Difference Equation With MatLab

<u>Example</u>

$$y(n) - y(n-1) + 0.9\,y(n-2) = x(n) \quad \forall n \sum_{K=0}^{M} a_K\,y(n-k) = \sum_{o}^{N} b_K\,x(n-1)$$

  a) Calculate and plot $h(n)$ for $n = -20, ..., 120$

  b) Calculate and plot the unit step response

  c) Is the system stable?

<u>Solution</u>

a) $a = [1, -1, 0.9], \quad b = [1];$

    $n = [-20 : 120];$

    $n_o = 0; \; x = [(n - n_o) = 0];$ % creates impulse at $\underline{0}$

    $h =$ filter *(b, a, x);*

    stem *(n, h);* title ('Impulse Response'); ylabel ('*h (n)*');

    xlabel ('*n*');

b) $x = [(n - n_o) >= 0];$

    $s =$ filter *(b, a, x);*

    stem *(n, s)* title ('Unit Step Response');

C) As can be seen *h(n)* is decaying $\rightarrow \Sigma|h(n)|$ can be determined by

    $\text{sum}(abs(h)) = 14.87 < \infty$

    Alternative method:

        $z =$ roots *(a);* (characteristic roots)

      magz =abs(z)

      $magz = \begin{cases} 0.9487 \\ 0.9487 \end{cases} \quad < 1 \rightarrow$ stable

Z-Transform (Chapter 3)

Z-Transform plays the same role as the Laplase Transform for CT signals. It is defined

as $x(z) = \sum\limits_{n=-\infty}^{+\infty} x(n)z^{-n}$ $x(n) \underset{Z}{\Longleftrightarrow} x(z)$

x(z) exists only for values that this power series converges.

The Region of Convergence (ROC) is the set of all values of *z*, where *x(z)* attains a finite

value.

Example 1

$x_1(n) = \{\underset{\uparrow}{1}, 2, 5, 7, 0, 1\}$

$X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$ ROC: entire *z* plane except *z = o*

$x_2(n) = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

$X_2(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3}$ ROC entire *z* plane except *z = 0, ∞*

Conclusion 1– In order to uniquely define *x(n)* from x(z), we have to know ROC.

Conclusion 2 – For a finite duration signal, ROC = entire *z* plane except *z = 0/∞*
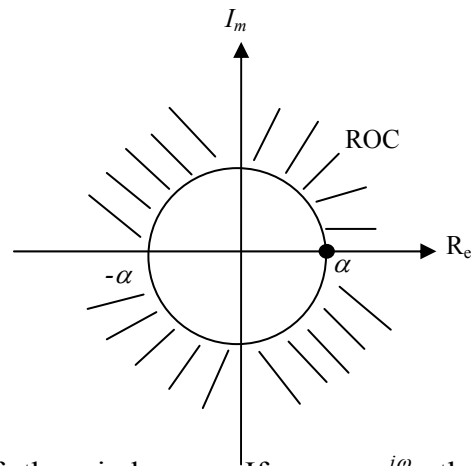
Exponential Signals

Example 1

$x(n) = \begin{cases} \alpha^n & n \geq 0 \\ 0 & n < 0 \end{cases}$

$X(z) = \sum\limits_{n=o}^{\infty} \alpha^n z^{-n} = \sum\limits_{n=o}^{\infty} \left(\alpha z^{-1}\right)^n$

If $\left|\alpha z^{-1}\right| < 1$ or $|z| > |\alpha|$, then $X(z) = \dfrac{1}{1 - \alpha z^{-1}}$.

In general, $z = re^{j\omega} \rightarrow |z| = |r| > |\alpha|$ outside of the circle α. If $z = re^{j\omega}$, then

$X(z) = \sum\limits_{n=-\infty}^{+\infty} x(n)r^{-n}e^{-j\omega n}$.
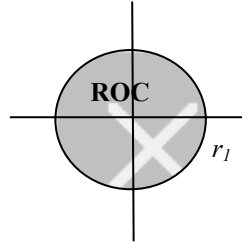
In the ROC of X(z), $|X(z)| < \infty$

But $|X(z)| = \left| \sum_{n=-\infty}^{+\infty} x(n) r^{-n} e^{-j\omega n} \right| \leq \sum_{-\infty}^{+\infty} \left| x(n) e^{-j\omega n} r^{-n} \right| = \sum_{-\infty}^{+\infty} \left| x(n) r^{-n} \right| < \infty$

Therefore, $|x(z)|$ is finite if $x(n)r^{-n}$ is absolutely summable. In order to find the range of r, we rewrite the above as:
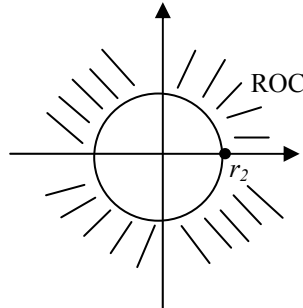
$$|x(z)| \leq \sum_{-\infty}^{-1} \left| x(n) r^{-n} \right| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right| = \sum_{n=1}^{\infty} \left| x(-n) r^n \right| + \sum_{0}^{\infty} \left| \frac{x(n)}{r^n} \right|$$

The convergence of the first term means there must exist values of $r_1 < \infty$ such that

$$\sum_{1}^{\infty} \left| x(-n) r_1^n \right| < \infty$$

The second term implies that there exists values $o < r_2 < \infty$ that $\sum_{0}^{\infty} \left| \frac{x(n)}{r_2^n} \right| < \infty$

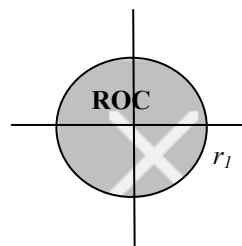Combing the two, therefore ROC is a ring $0 < r_2 < r < r_1 < \infty$. If $r_2 > r_1$, then there is no ROC.

Example 2

$$x(n) = -\alpha^n U(-n-1) = \begin{cases} 0 & n \geq 0 \\ -\alpha^n & n \leq -1 \end{cases}$$

$X(z) = \sum_{-\infty}^{-1} \left(-\alpha^n \right) z^{-n} = - \sum_{n=1}^{\infty} \left(\alpha^{-1} z \right)^n$, if $\left| \alpha^{-1} z \right| < 1$ then $X(z) = \frac{-\alpha^{-1} z}{1 - \alpha^{-1} z} = \frac{1}{1 - \alpha z^{-1}}$

As can be seen, $X(z)$ of this example is the same as the one in Example 1, but their ROC

are different. Here, $\left|\alpha^{-1}z\right| < 1 \rightarrow \left|\dfrac{z}{\alpha}\right| \leq \dfrac{|z|}{|\alpha|} < 1 \rightarrow |z| < |\alpha|$

Comparing Example 1 and 2:

1) We have to know ROC in order to define $x(n)$ from $X(z)$ uniquely.

2) ROC of a causal signal is the exterior of a circle of some radius while the ROC of a non-causal signal is the anterior of a circle.

Example 3

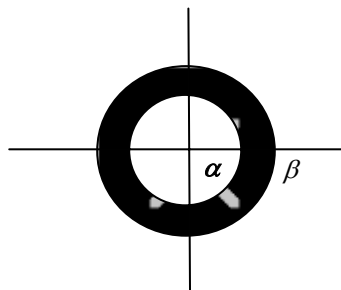$$x(n) = \alpha^n u(n) + \beta^n U(-n-1)$$

$$\downarrow Z$$

$$X(z) = \frac{1}{1-\alpha z^{-1}} - \frac{1}{1-\beta z^{-1}}$$

but each of these terms exists if

$$|z| > |\alpha| \text{ and } |z| < |\beta|$$

Therefore, $X(z)$ exists if $\exists \alpha$ and β such that $|\alpha| < |\beta|$. Then $\alpha < |z| < \beta$ is the ROC

Conclusion

ROC for an infinite two-sided duration signal is a ring.

Read the summary on page 159 of the textbook.