

Sections 4.4.5 and 4.2.6

Relationship of Z -Transform and Fourier Transform

$$X(z) = \sum_{n=-\infty}^{+\infty} x(n)z^{-n} \quad \text{ROC: } r_2 < |z| < r_1$$

$$\text{let } z = re^{j\omega} \text{ then } X(z)\Big|_{z=re^{j\omega}} = \sum_{n=-\infty}^{+\infty} x(n)r^{-n}e^{-j\omega n} \quad \text{and } X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n} \equiv X(\omega)$$

Therefore, if $z = e^{j\omega}$ is not within ROC, then $X(\omega)$ doesn't exist. There are cases that $X(z)$ exists

such as $a^n u(n)$, $|a| > 1$, because we can find an r such that $\sum_{-\infty}^{+\infty} |x(n)r^{-n}| < \infty$, but its $X(\omega)$ doesn't

exist because $\sum |x(n)|$ is not finite when $|a| > 1$. Note that in this example, ROC doesn't include $z = 1$ since ROC will be $|z| > r > |a| > 1$.

On the other hand, there are cases that $X(\omega)$ exists on a weaker condition that the signal's energy

is finite like $x(n) = \frac{\sin \omega_c n}{\pi n}$ but it doesn't have a $X(z)$. Therefore, if $X(z)$ exists and if its ROC

includes unit circle, then $X(\omega)$ exists too, while the other side around is not true.

If the system function, $H(z)$, converges on the unit circle, we can obtain the frequency response of the system by evaluating $H(z)$ on the unit circle.

$$H(\omega) = H(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{+\infty} h(n)e^{-j\omega n}$$

If $H(z)$ can be written as $H(z) = \frac{B(z)}{A(z)}$ then

$$H(\omega) = \frac{B(\omega)}{A(\omega)} = \frac{\sum_{k=0}^M b_k e^{-j\omega k}}{1 + \sum_{k=1}^N a_k e^{-j\omega k}} = b_0 \frac{\prod_{k=1}^M (1 - z_k e^{-j\omega})}{\prod_{k=1}^N (1 - p_k e^{-j\omega})} \quad a_k \text{ and } b_k \text{ are real, but } z_k \text{ and } p_k \text{ can be}$$

complex.

Power Spectrum is defined as: $|H(\omega)|^2 = H(\omega) \cdot H^*(\omega)$

$$H^*(\omega) = b_o \frac{\prod_{k=1}^M (1 - z_k^* e^{j\omega})}{\prod_{k=1}^N (1 - p_k^* e^{j\omega})} \equiv H^*\left(\frac{1}{z^*}\right) = b_o \frac{\prod_{k=1}^M (1 - z_k^* z)}{\prod_{k=1}^N (1 - p_k^* z)}$$

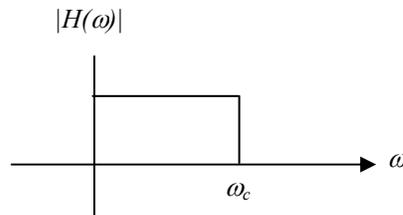
when $h(n)$ is real, then complex z_k and p_k occur in complex-conjugate pairs. Then $H^*\left(\frac{1}{z^*}\right) = H(z^{-1})$ or equivalently $H^*(\omega) = H(-\omega)$.

LTI Systems as Frequency Selective Filters

$H(\omega)$ acts as a weighting function or a spectral shaping function as $Y(\omega) = H(\omega) \cdot X(\omega)$. From this point of view, $H(\omega)$ is a filter. An ideal filter has a constant gain in pass-band and is zero in stop-band and also has a linear phase response. $\theta(\omega) = -\omega n_0$ -linear characteristics within pass-band.

$\tau_g = -\frac{d\theta(\omega)}{d\omega}$ is called the “group delay” of the filter. $\tau_g(\omega)$ is the time delay that a signal component of frequency ω , undergoes as it passes from the input to the output of the system. Obviously when $\tau_g(\omega) = \text{constant}$, then all the components have the same delay.

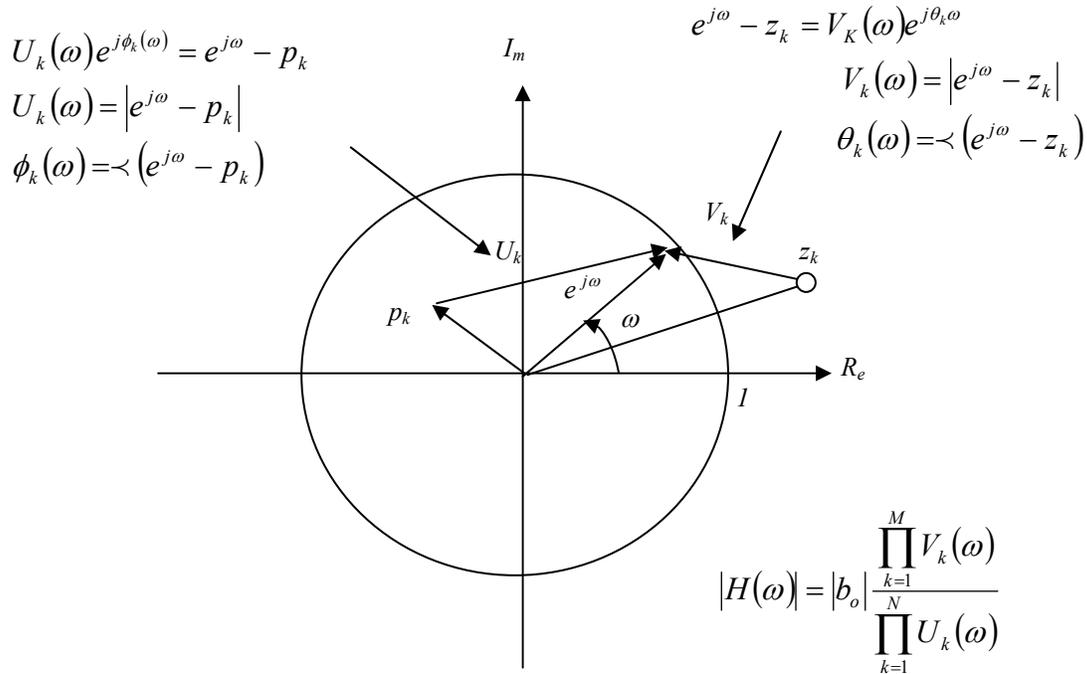
An example of ideal low-pass filter is:



Therefore, $h(n) = \frac{\sin \omega_c \pi n}{\pi n}$ but is not a casual signal and is not absolutely summable and is

also unstable. However, its frequency response can be very closely approximated by some realizable filter. You have seen how the location of poles and zeros changes the frequency response. Now lets have a graphical view of their location and type of filter.

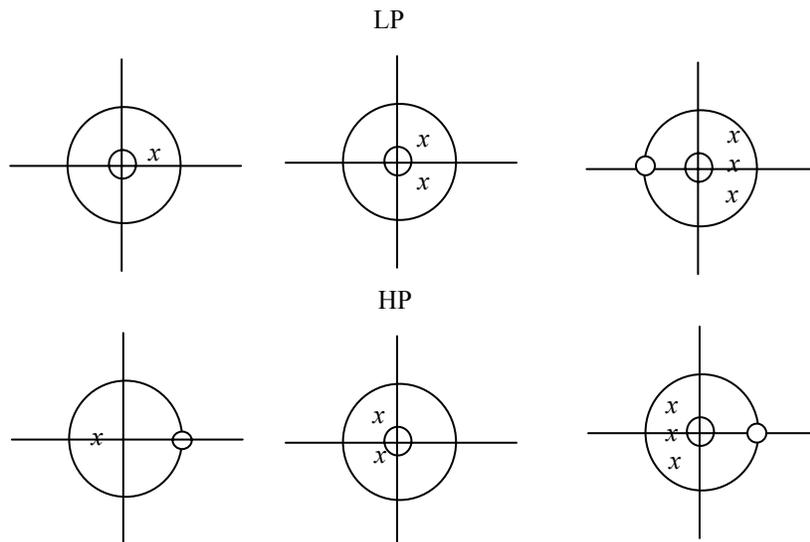
$$H(\omega) = b_o \frac{\prod_{k=1}^M (1 - z_k e^{-j\omega})}{\prod_{k=1}^N (1 - p_k e^{-j\omega})} = b_o e^{j\omega(N-M)} \frac{\prod_{k=1}^M (e^{j\omega} - z_k)}{\prod_{k=1}^N (e^{j\omega} - p_k)} = b_o e^{j\omega(N-M)} \frac{\prod_{k=1}^M V_k(\omega) e^{j\theta_k(\omega)}}{\prod_{k=1}^N U_k(\omega) e^{j\phi_k(\omega)}}$$



- 1) If a zero is on the unit circle at $\omega_o = \angle z_k$, then $V_k|_{\omega=\omega_o} = 0 \Rightarrow |H(\omega_o)| = 0$.
- 2) If a pole is on the unit circle at $\omega_o = \angle p_k$, then $U_k|_{\omega=\omega_o} = 0 \Rightarrow |H(\omega_o)| = \infty$.

From 1 and 2 it is clear that the presence of a zero close to the unit circle, makes $|H(\omega)|$ to be small at the frequencies close to that point and on the other hand, the presence of a pole close to the unit circle, causes $|H(\omega)|$ to be large at the frequencies close to that point.

Lets look at $H(z) = \frac{1-a}{1-az^{-1}}$ $p_1 = a$ $z_1 = 0$. So if a is real ($\angle p_1 = 0$) and close to unit circle, then $|H(\omega)|$ will be maximum at zero frequencies. Now if you add a zero on the unit circle, but with $\omega - \pi$, it also attenuates the $|H(\omega)|$ more at high frequencies. Therefore, we can say what kind of filter it is by just looking at zero-poles locations.



Example of a Band-Pass Filter

Design a 2-pole band-pass filter with center frequency $\omega = \frac{\pi}{2}$ and 2-zeros at $\omega = 0, \pi$, also

$|H(\omega)|_{\omega=\frac{4\pi}{9}} = \frac{1}{\sqrt{2}}$. Lets have the zeros on the unit circle: $z_1 = e^{j0} = 1, z_2 = e^{+j\pi} = -1$. If we want

the filter coefficients to be real, the complex poles must be complex conjugate. So let

$$\begin{aligned} p_1 &= re^{j\pi/2} & p_2 &= re^{-j\pi/2} \\ &= jr & &= -jr \end{aligned}$$

$\rightarrow H(z) = G \frac{(z-1)(z+1)}{(z-jr)(z+jr)}$ will choose G such that

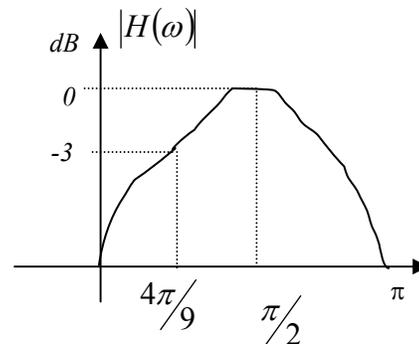
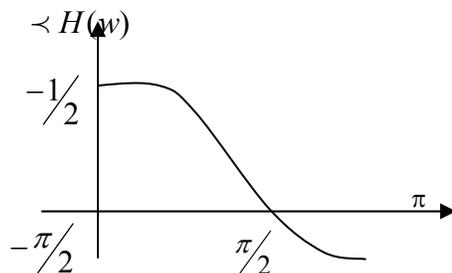
$$\left| H\left(\frac{\pi}{2}\right) \right| = 1 \rightarrow H(z) \Big|_{z=e^{j\pi/2}} = G \frac{z^2-1}{z^2+r^2} \Big|_{z=j} = G \frac{-2}{-1+r^2} = 1 \rightarrow G = \frac{r^2-1}{2}$$

To determine the value r , we should use the corner frequency or the 3db frequency.

$$\left\{ |H(\omega)|_{\omega=\frac{4\pi}{9}} \right\}^2 = \left\{ |H(\omega)|_{z=e^{j4\pi/9}} \right\}^2 = \frac{1}{2}$$

$$\underbrace{\frac{r^2-1}{2}}_G \cdot \left| \frac{\left(\cos \frac{8\pi}{9} - 1 \right) + j \sin \frac{8\pi}{9}}{\left(\cos \frac{8\pi}{9} + r^2 \right) + j \sin \frac{8\pi}{9}} \right|^2 = \frac{r^2-1}{2} \frac{2-2\cos \frac{8\pi}{9}}{1+r^4+2r^2\cos \frac{8\pi}{9}} = \frac{1}{2} \rightarrow r^2 = 0.7$$

$$H(z) = 0.15 \frac{1-z^{-2}}{1+0.7z^{-2}}$$



Converting LP to HP Filters

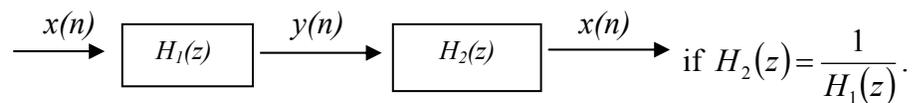
$$H_{HP}(\omega) = H_{LP}(\omega - \pi) \rightarrow h_{HP}(n) = e^{j\pi n} h_{LP}(n) = (-1)^n h_{LP}(n)$$

All-Pass Filters

If $|H(\omega)| = 1$ for $0 < \omega < \pi \Rightarrow H(\omega)$ is an all-pass system like a delay system. $H(z) = z^{-k}$. In an all-pass system, if z_0 is a pole, then $\frac{1}{z_0}$ is a zero. Its main application is in phase equalizer to compensate for poor phase characteristics to produce an overall linear phase response.

Read all of Section 4.5.

Invertibility of LIT Systems



If $H_1(z) = \frac{B(z)}{A(z)}$, then $H_2(z) = \frac{A(z)}{B(z)}$ meaning that the zeros of $H_1(z)$ are the poles of $H_2(z)$. Is

every system invertible?

If an invertible system cannot be expressed by z-transform, we may find the inverse at the system by convolution of the two $h_1(n)$ and $h_2(n)$.

$$\sum_{k=0}^n h_1(k)h_2(n-k) = \delta(n) \quad h_2(n) = 0 \text{ for } n < 0. \text{ For } n = 0 \rightarrow h_2(0) = \frac{1}{h_1(0)}.$$

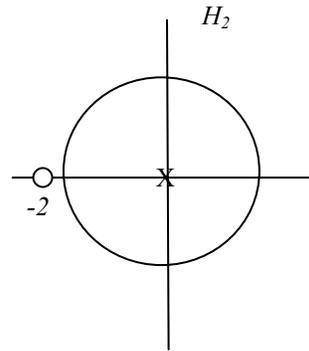
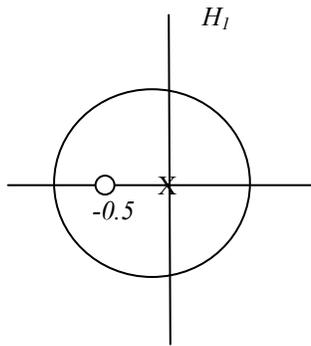
For $n \geq 1$, $h_2(n) = \frac{-1}{h_1(0)} \sum_{k=1}^n h_1(k)h_2(n-k)$. You can prove these easily by expanding the convolution sum.

This method does not work if $h_1(0) = 0$ but this can be easily resolved by introducing a delay.

Minimum Phase – Maximum Phase Systems

Consider these two systems:
$$\begin{cases} H_1(z) = z^{-1} \left(z + \frac{1}{2} \right) = 1 + \frac{1}{2} z^{-1} \\ H_2(z) = \frac{1}{2} + z^{-1} = z^{-1} \left(\frac{1}{2} z + 1 \right) \end{cases}$$

Both are all-zeros systems.



$$H_1(\omega) = 1 + \frac{1}{2}e^{-j\omega} = \left(1 + \frac{1}{2}\cos\omega\right) - \frac{j}{2}\sin\omega$$

$$|H_1(\omega)|^2 = \left(1 + \frac{1}{2}\cos\omega\right)^2 + \frac{1}{4}\sin^2\omega = \frac{5}{4} + \cos\omega$$

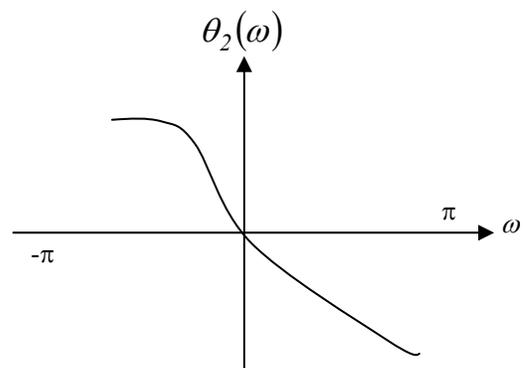
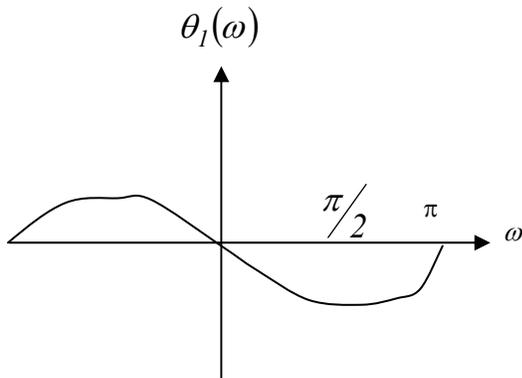
$$H_2(\omega) = \frac{1}{2} + e^{-j\omega} = \left(\frac{1}{2} + \cos\omega\right) - j\sin\omega$$

$$|H_2(\omega)|^2 = \frac{5}{4} + \cos\omega$$

You can plot and check these with
`[H, W] = freqz(b,a,1024,'whole')`
 Then plot `(w, angle(H))`

$$\rightarrow |H_1(\omega)| = |H_2(\omega)| \text{ because } z_2 = \frac{1}{z_1} \text{ reciprocal but } \theta_1(\omega) = \tan^{-1} \frac{\sin\omega}{\frac{1}{2} + \cos\omega} - \omega$$

$$\theta_2(\omega) = \tan^{-1} \frac{\sin\omega}{2 + \cos\omega} - \omega$$

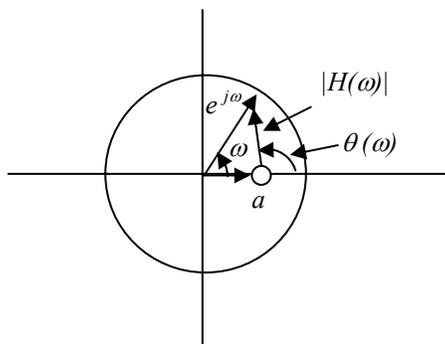


As ω goes from $0 \rightarrow \pi$, the phase change of $\theta_1(\omega)$ is zero but for $\theta_2(\omega)$ is π . Therefore H1 has the minimum phase while H2 has the maximum phase change. So a Min-Phase system has zeros all inside the unit circle. On the other hand, all zeros of a Max-Phase system are outside the unit circle. Min-phase \equiv Minimum delay.

Showing Minimum Phase Graphically:

$$H(z) = 1 - az^{-1} = \frac{z-a}{z} \Rightarrow H(\omega) = \frac{e^{j\omega} - a}{e^{j\omega}} \quad z_1 = a, P_1 = 0$$

Case I: $|a| < 1 \rightarrow H(\omega) = \theta(\omega) - \omega$



In this case, if ω changes from zero to π ,

$\rightarrow H(\omega)$ changes from zero to zero.

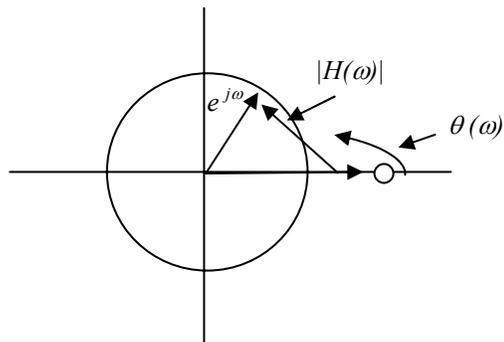
At $\omega = 0 \rightarrow H(\omega) = \theta(0) - 0 = 0$

At $\omega = \pi \rightarrow H(\omega) = \theta(\pi) - \pi = 0$

\rightarrow Net change = 0

\rightarrow Minimum Phase System

Case II: $|a| > 1$



At $\omega = 0 \rightarrow H(\omega) = \pi - 0 = \pi$

At $\omega = \pi \rightarrow H(\omega) = \pi - \pi = 0$

Therefore, in this case, net charge = π and the system is therefore, a Max-Phase system.

So in general, if a system has M zeros outside the unit circle, its net phase change will be $M \cdot \pi$ over the range $(0, \pi)$, while the net phase change for the zeros or poles inside the unit circle is zero.

Phase Compensation

Any casual and stable system can be decomposed to a Min-Phase system cascaded with an all-pass system.

$$H(z) = H_{\min}(z) \cdot H_{ap}(z)$$

$$\text{Lets say } H(z) = \frac{1+3z^{-1}}{1+\frac{1}{2}z^{-1}} = \frac{z+3}{z+\frac{1}{2}} \Rightarrow \begin{cases} z_1 = -3 \\ p_1 = -\frac{1}{2} \end{cases}$$

We need to choose the appropriate H_{ap} to reflect the zero that is outside the unit circle to inside. So H_{ap} should have a zero at $z_1 = -3$ and since it is an all-pass system, it must have a pole at

$$\frac{1}{z_1} = \frac{-1}{3}. \text{ Therefore, } H_{ap} = \frac{z+3}{z+\frac{1}{3}} = \frac{1+3z^{-1}}{1+\frac{1}{3}z^{-1}} \rightarrow H_{\min} = \frac{H(z)}{H_{ap}(z)} = \frac{z+\frac{1}{3}}{z+\frac{1}{2}} = \frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{2}z^{-1}}$$

Another Example

$$H(z) = \frac{\left(1 + \frac{3}{2}e^{j\pi/4}z^{-1}\right)\left(1 + \frac{3}{2}e^{-j\pi/4}z^{-1}\right)}{\left(1 - \frac{1}{3}z^{-1}\right)} \Rightarrow \begin{cases} z_{1,2} = \frac{-3}{2}e^{\pm j\pi/4} \\ p_1 = \frac{1}{3}, p_2 = 0 \end{cases}$$

Note that $H_{ap}(z)$ in these cases is also a Max-Phase System

$$H_{ap}(z) = \frac{\left(z + \frac{3}{2}e^{j\pi/4}\right)\left(z + \frac{3}{2}e^{-j\pi/4}\right)}{\left(z + \frac{2}{3}e^{j\pi/4}\right)\left(z + \frac{2}{3}e^{-j\pi/4}\right)}$$

$$\rightarrow H_{\min}(z) = \frac{\left(z + \frac{2}{3}e^{j\pi/4}\right)\left(z + \frac{2}{3}e^{-j\pi/4}\right)}{z\left(z - \frac{1}{3}\right)} = \frac{\left(1 + \frac{2}{3}e^{j\pi/4}z^{-1}\right)\left(1 + \frac{2}{3}e^{-j\pi/4}z^{-1}\right)}{1 - \frac{1}{3}z^{-1}}$$

Verify these with MATLAB

Min-Phase System Properties

1. Min-Phase System has the smallest group delay.

$$\text{Proof: } H(z) = H_{\min}(z) \cdot H_{ap}(z)$$

$$\tau_g(\omega) = \tau_{g\min}(\omega) + \tau_{gap}(\omega)$$

$$\text{Since } \tau_{gap}(\omega) \geq 0 \text{ for } 0 \leq \omega \leq \pi \Rightarrow \tau_g(\omega) \geq \tau_{g\min}(\omega)$$

2. The Min-Phase System has the largest partial energy. Partial Energy of a system is defined

$$\text{as } E(n) = \sum_{k=0}^n |h(k)|^2$$

It can be shown that among all systems having the same $|H(\omega)|$ and same total energy $E(\infty) = \sum_{k=0}^{\infty} |h(k)|^2$, the Min-Phase System has the largest partial energy. In particular, it can be also concluded that $|h_{MP}(0)| \geq |h(0)|$.

Use initial value theorem and prove this as assignment. Look also to following problems in Chapter 4: 6(a,c), 8, 28, 30, 51, 59 and 100.