Welch Method

Welch Method is in fact the modified Bartlet method by the following two modifications:

1. Segments have an overlap
2. Applying a window rather than ideal Rectangle window to each segment.

Therefore, $x_i(n) \equiv$ data samples in each segment $= x(n + iD)$,

where $n = 0, \ldots.., M-1$ $\quad$ M = length of each segment

$i = 0, \ldots.., K-1$ $\quad$ K = number of segments

To do segmentation in Matlab:

For $i = 1: K$

$x_i(n) = x\left(1 + (i-1)M - (i-1)\frac{M}{2} : i \cdot M - (i-1) \cdot \frac{M}{2}\right)$; with 50% overlap $M = D/2$ end;

If $D = M$, then there is no overlap. If $D = \frac{M}{2}$, there is a 50% overlap between the successive segments. Also, there is a window for segmentation called Data Window $w_d(n)$ is a window such that $\int_{-1/2}^{1/2} W_d(f) \cdot df = 1 = w_d(0)$.

There are many different choices for $w_d(n)$ such as Hanning, Hamming, Blackman, Parzen, etc., windows.

The power spectrum estimation in Welch Method is then defined as $\widetilde{P}_x^{\,w}(\omega) = \frac{1}{K}\sum_{i=0}^{K-1} \widetilde{P}_x^{\,i}(\omega)$.

where $\widetilde{P}_x^{\,i}(\omega) = \frac{1}{M \cdot U}\left|\sum_{n=0}^{M-1} x_i(n) w_d(n) e^{-j\omega n}\right|^2$, $i = 0, \ldots., K-1$ and $U = \frac{1}{M}\sum_{n=0}^{M-1} w_d^2(n)$ is the normalization factor to result in $\int_{-1/2}^{1/2} W_d(f) df = 1$

Now, $E\left\{\widetilde{P}_x^{\,w}(\omega)\right\} = \frac{1}{K}\sum_{i=0}^{K-1} E\left\{\widetilde{P}_x^{\,i}(\omega)\right\} = E\left\{\widetilde{P}_x^{\,i}(\omega)\right\}$ and

$$E\left\{\widetilde{P}_x^{\,i}(\omega)\right\} = \frac{1}{MU}\sum_{n=0}^{M-1}\sum_{m=0}^{M-1} w_d(n) w^*{}_d(m) E\{x_i(n) x_i^*(m)\} e^{-j\omega(n-m)}$$

$$= \frac{1}{MU}\sum_{n=0}^{M-1}\sum_{m=0}^{M-1} w_d(n) w^*{}_d(m) \gamma_{xx}(n-m) e^{-j\omega(n-M)}$$

but, $\gamma_x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Gamma_x(\alpha) e^{j\alpha n} d\alpha$

$$\Rightarrow E\left\{\widetilde{P}_x^i(\omega)\right\} = \frac{1}{MU}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma_x(\alpha)\left[\sum_{m=0}^{M-1}\sum_{n=0}^{M-1} w_d(n)w^*{}_d(m)e^{-j(n-m)(\omega-\alpha)}\right]d\alpha$$

But $S_{wd}(\omega-\alpha) \equiv$ PSD of w_d and $S_{wd}(\omega) = \frac{1}{MU}\left|\sum_{n=0}^{M-1} w_d(n)e^{-j\omega n}\right|^2$

$$\Rightarrow E\left\{\widetilde{P}_x^i(\omega)\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma_x(\alpha)S_{Wd}(\omega-\alpha)d\alpha = \Gamma_x(\omega) * S_{Wd}(\omega)$$

So the bias of PSD with the Welch method depends on the shape of data window. You can expect the narrower the main lobe of $S_{Wd}(\omega)$, the smaller the bias. But again as usual, the lesser bias may cause a larger variance of the estimator.

$$\mathrm{var}\{P_x(\omega)\} = \frac{1}{K}\mathrm{var}\left\{\widetilde{P}_x^i(\omega)\right\} \approx \begin{cases} \frac{1}{K}\Gamma_x^2(\omega) & \textit{with no overlap} \\ \frac{9}{8K}\Gamma_x^2(\omega) & \textit{with } 50\% \textit{ overlap} \end{cases}$$

** Note that the K in case of no overlap in above is not the same as the one in the case of 50% overlap.

Blackman and Tukey Method (Smoothing Periodogram)

In this method, you calculate autocorrelation, window it and then take FFT to estimate power spectrum.

$$P_x^{BT}(\omega) = \sum_{m=-M+1}^{M-1}\gamma_{xx}(m)\cdot w(m)\cdot e^{-j\omega m}$$ $w(m)$ has the length of $2M-1$.

$$P_x^{BT}(\omega) = FFT\{\gamma_{xx}(m)\cdot w(m)\} = \left[\widetilde{\Gamma}_x(\omega) * W(\omega)\right]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}P_x(\alpha)W(\omega-\alpha)d\alpha$$, where P_x is the periodogram.

The window is chosen to be symmetric and that $W(\omega) \geq 0$ for $|\omega| < \pi$ so that the power spectrum doesn't become negative.

Let's check its bias and variance:

$$E\left\{P_x^{BT}(\omega)\right\} = E\left\{\widetilde{\Gamma}_x(\omega) * W(\omega)\right\} = E\left\{\widetilde{\Gamma}_x(\omega)\right\} * W(\omega)$$

but $\Gamma_x(\omega)$ is in fact the periodogram and recalling that the expected value of the periodogram is $\sum_{\ell=-N+1}^{N-1}\left(\underbrace{1-\frac{|\ell|}{N}}_{w^B(\ell)}\right)\gamma(\ell)e^{-j\omega\ell}$ which is the Fourier transform of $w^B(\ell)\cdot\gamma(\ell)$. Therefore, it is equivalent to $W^B(\omega)*\Gamma(\omega)$ where $W^B(\omega)=FFT\left(1-\frac{|\ell|}{N}\right)=\frac{1}{N}\left(\frac{\sin\pi f N}{\sin\pi f}\right)^2$

$$\rightarrow E\{P_x^{BT}(\omega)\}=\Gamma_x(\omega)*W^B(\omega)*W(\omega)$$

If we select $w(n)$ such that $M << N$, then it means that $w(n)$ is much narrower than $w^B(\ell)$. Therefore, $W(\omega)$ is much wider than $W^B(\omega)$. Hence, $W^B(\omega) * W(\omega) \approx W(\omega)$.

So, the bias of the Blackman-Tukey estimator is approximated as

$$E\{P_x^{BT}(\omega)\}\approx\Gamma_x(\omega)*W(\omega)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma_x(\theta)W(\omega-\theta)d\theta.$$

The variance of Blackman-Tukey estimator has been derived as:

$$\text{var}\{P_x^{BT}(\omega)\}\approx\frac{1}{N}\frac{1}{2\pi}\int_{-\pi}^{\pi}\Gamma_x^2(\alpha)W^2(\omega-\alpha)d\alpha$$

If $W(\omega)$ is narrow compared to the $\Gamma_x(\omega)$, then

$$\begin{aligned}\text{var}\{P_x^{BT}\}&\approx\Gamma_x^2(\omega)\left[\frac{1}{N}\frac{1}{2\pi}\int_{-\pi}^{\pi}W^2(\theta)d\theta\right]\\&=\underbrace{\Gamma_x^2(\omega)}_{P_x^2(\omega)}\cdot\left[\frac{1}{N}\sum_{n=-M+1}^{M-1}w^2(n)\right]=\frac{1}{N}\Gamma_x^2(\omega)\cdot E_w\end{aligned},$$

where E_w is the energy of the window.

<u>Window Selection for Blackman-Tukey Method</u>

For a satisfactory $P_x(\omega)$, the bias should be small, and also the variance of $P_x(\omega)$ must be small compared to $P_x^2(\omega)$. Equivalently, the variance ratio: $\beta=\frac{\text{var}\{P_x^{BT}(\omega)\}}{P_x^{BT^2}(\omega)}<<1$.

This is the case if energy of the window $E_w\approx N\beta<<N$. It means that $w(n)$ must take significant values only in the interval $(-M, M)$ such that $M << N$. We assume that $|w(0)|\leq 1$ for $|n| < M$ and outside of $\pm M$ is zero (not necessary but a convenient assumption). Therefore,

$E_w \le 2M \Rightarrow \beta \le \dfrac{2M}{N} \Rightarrow$ to satisfy the variance to be small, M has to be much smaller than N. With M determined to satisfy this condition, then the shape of window is chosen to minimize the bias. So, the most important factor is the width of window, not the shape.

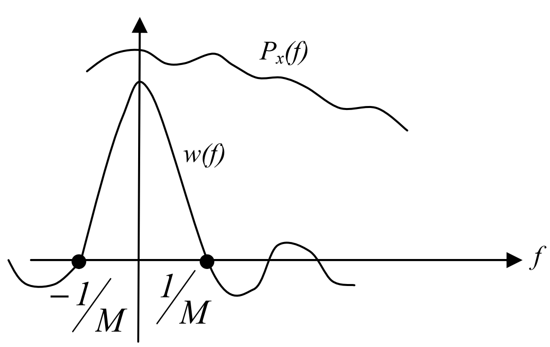
Summary of window's properties:

$$\begin{cases} w(0)=1 \\ w(n)=w(-n) \\ w(n)=0 \; for \, |n| \ge M, M < N \end{cases} \equiv \begin{cases} \int_{-1/2}^{1/2} w(f)df = w(0) = 1 \\ w(f)=w(-f) \\ w(f) \end{cases}$$

It is a slit with base width of order 2/M

We said that the width is more important. However, one should keep in mind that there is a trade off between the bias and variance of the spectral estimation. Another sensible approach is to compromise by making the $MSE = \mathrm{var}\{\tilde{P}_x(f) + B^2\}$, where $B = E\{\tilde{P}_x(f)\} - P_x(f)$ as small as possible.

The exact nature of the compromise, which has to be made, will depend on the degree of smoothness of the spectrum $P_x(f)$. For example, if $P_x(f)$ is very smooth, then the variance may be reduced by using a wide window without having serious effect on the bias. In particular, if $P_x(f)$ is smooth over the range of $\pm\dfrac{1}{M}$, then $E\{\tilde{P}_x(f)\} \approx P_x(f)\int_{-1/2}^{1/2} w(f)df = P_x(f)$



Hence, when the spectrum is sufficiently smooth, a virtually unbiased estimator can be obtained even though the spectral window has been made wide (in frequency domain) to reduce the variance.

Windows Carpentry

There are several windows. Some of the most popular ones are:

Rectangular Window: $$w^R(n) = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{else} \end{cases}$$

Bartlett (Triangular) Window $$w^B(n)=\begin{cases} 2n/\mathrm{M} & 0\le n\le M/2 \\ 2-2n/\mathrm{M} & M/2<n\le M \\ 0 & \text{else}\end{cases}$$

Hanning Window $$w^{Han}(n)=\begin{cases} 0.5-0.5\cos(2\pi \mathrm{n}/\mathrm{M}) & 0\le n\le M \\ 0 & \text{else}\end{cases}$$

Hamming Window $$w^{Ham}(n)=\begin{cases} 0.54-0.46\cos(2\pi \mathrm{n}/\mathrm{M}) & 0\le n\le M \\ 0 & \text{else}\end{cases}$$

Blackman Window $$w^{Bl}(n)=\begin{cases} 0.42-0.5\cos(2\pi \mathrm{n}/\mathrm{M})+0.08\cos(4\pi \mathrm{n}/\mathrm{M}) & 0\le n\le M \\ 0 & \text{else}\end{cases}$$

The Fourier Transform of the above windows (in continuos forms though) are as the followings. Note that these formulas slightly changes for the case the widnows are discrete.

Rectangle Window $$W^R(f)=2M\frac{\sin 2\pi fM}{2\pi fM}$$

Bartlet Window *(=* $w^R * w^R$*)* $$W^B(f)=M\left(\frac{\sin \pi fM}{\pi fM}\right)^2$$

Parzen Window *(=*$w^B * w^B$*)* $$W^P(f)=\frac{3}{4}M\left(\frac{\sin \pi f\, M/2}{\pi f\, M/2}\right)^4$$

Tukey Window $$W^T(f)=M\left(\frac{\sin 2\pi fM}{2\pi fM}\frac{1}{1-(2\pi fM)^2}\right)$$

w^R, w^B, and w^P are almost $\approx\left(\dfrac{\sin 2\pi f\, M/n}{2\pi f\, M/n}\right)^n$

$n = 1, 2, 4$ for w^R, w^B, and w^P, respectively.

As n increases, you'll see that the log window tends to shape like Normal Curve.

w^R has the narrowest main lobe but big side lobes. By increasing n, the magnitude of these side lobes decreases.

Performance Characteristics of Non-Parametric Methods

Lets define a measure of quality $Q = \frac{\lfloor E\{\widetilde{P}_x(f)\}\rfloor^2}{\operatorname{var}\{\widetilde{P}_x(f)\}}$

For periodogram: $E\{\hat{P}_x^p(\omega)\} = \sum_{\ell=-N+1}^{N-1}\left(1-\frac{|\ell|}{N}\right)\gamma(\ell)e^{-j\omega\ell}$

and $\operatorname{var}\{\widetilde{P}_x^P(\omega)\} = \Gamma_{xx}^2(\omega)\left[1+\frac{1}{N}\left(\frac{\sin \pi f N}{\sin \pi f}\right)^2\right]$

as N $\rightarrow \infty$ $\qquad E\{\widetilde{P}_x^P(\omega)\} \rightarrow P_x(\omega)$

and $\qquad \operatorname{var}\{\widetilde{P}_x^P(\omega)\} \rightarrow P_x^2(\omega)$

Therefore, $Q^P \xrightarrow[N\rightarrow\infty]{} 1$, which is not good because it means it is independent of N and increasing of N is not going to improve the variance and quality.

For Bartlet Method:

As N and M increases, the quality factor goes to K that is the number of segments:

$$\underbrace{E\{\widetilde{P}_x^B(\omega)\}}_{N\rightarrow\infty} \rightarrow \Gamma_x(\omega)\frac{1}{2\pi}\int_{-\pi}^{\pi} W^B(\omega)d\omega = \Gamma_x(\omega)w^B(0) = \Gamma_x(\omega)$$

$$\underbrace{\operatorname{var}\{\widetilde{P}_x^B(\omega)\}}_{N\rightarrow\infty} \rightarrow \frac{1}{K}\Gamma^2{}_x(\omega)$$

Hence the quality factor becomes $Q^B_{N\rightarrow\infty} \rightarrow K = \frac{N}{M}$

The frequency resolution of the Barlet estimate, measured by taking the 3dB width of the main lobe of the Rectangle Window is $\Delta f = \frac{0.9}{M} \Rightarrow Q^B = \frac{N}{0.9/\Delta f} = 1.1N.\Delta f$ while $Q^W = 1.39N.\Delta f$ with 50% overlap and $Q^{BT} = 2.34N.\Delta f$.

Note that N and Δf are appositely related.