# Lecture 5

**Calculating the Inverse Z-Transform**

$x(n) = \frac{1}{2\pi j}\oint X(z) z^{n-1} dz$ Three methods to calculate it:

1) Direct Method by contour integration
2) Expansion into a series of terms $z/z^{-1}$
3) Partial Fraction expansion and look-up table.

Cauchy-Residue Theorem

Let $f(z)$ be a function of the complex variable $z$ and $C$ be a closed path in the z-plane. If the derivative $\frac{d}{dz}f(z)$ exists on and inside the contour $C$ and if $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j}\oint_c \frac{f(z)}{(z-z_0)}dz = \begin{cases} f(z_0) & \text{if } z_0 \text{ is inside } C \\ 0 & \text{else} \end{cases}$$

More generally:

$$\frac{1}{2\pi j}\oint_c \frac{f(z)}{(z-z_0)^k}dz = \begin{cases} \frac{1}{(k-1)!} \frac{d^{k-1}f(z)}{dz^{k-1}}\Big|_{z=z_0} & \text{if } z_0 \text{ inside } C \\ 0 & \text{else} \end{cases}$$

RHS of the above equation is called ***residue of the pole*** $z_o$.

Now suppose the function can be written as $\frac{f(z)}{g(z)}$ where $f(z)$ has no poles inside $C$ and $g(z)$ is a polynomial with simple roots $z_1, z_2, \ldots, z_n$ inside $C$. Then,

$$\frac{1}{2\pi j}\oint_c \frac{f(z)}{g(z)}dz = \frac{1}{2\pi j}\oint_c \left[\sum_{i=1}^{n}\frac{A_i(z)}{z-z_i}\right]dz = \sum_{i=1}^{n}\frac{1}{2\pi j}\oint \frac{A_i(z)}{z-z_i}dz = \sum_{i=1}^{n} A_i(z_i),$$

where $A_i(z) = (z-z_i)\frac{f(z)}{g(z)}$.

$\Rightarrow x(n) = \frac{1}{2\pi j}\oint x(z) z^{n-1}dz = \sum_{i=1}^{N}(z-z_i)X(z)z^{n-1}\Big|_{z=z_i}$ = sum at residue of $x(z)z^{n-1}$ at $z = z_i$ and $N$ = number of poles.

**Example:** Problem 3.56 (C)

$$X(z)=\frac{z-a}{1-az} \qquad |z|>\frac{1}{|a|}$$

$$x(n)=\frac{1}{2\pi j}\oint_c \frac{z-a}{1-az}z^{n-1}dz=\frac{1}{2\pi j}\oint_c -\frac{1}{a}\frac{z-a}{z-1/a}\cdot z^{n-1}dz$$

$z=r>\frac{1}{|a|}$ This is the contour C.

Three cases:

1) For $n\geq 1$ then $f(z)=\frac{-1}{a}(z-a)z^{n-1}$ and the only pole inside $c$ is $\frac{1}{a}$. Therefore, for $n\geq 1$

$$x(n)=\frac{-1}{a}(z-a)z^{n-1}\Big|_{z=\frac{1}{a}}=\frac{-1}{a}\left(\frac{1}{a}-a\right)\left(\frac{1}{a}\right)^{n-1}$$

$$=\left(\frac{1}{a}\right)^{n-1}-\left(\frac{1}{a}\right)^{n+1}$$

2) For $n=o\Rightarrow x(n)=\frac{1}{2\pi j}\oint\frac{z-a}{z(1-az)}dz$

then, $f(z)=\frac{-1}{a}(z-a)$ and two poles (*0* and $\frac{1}{a}$) inside *C*. Therefore,

$$x(n)=\frac{-1}{a}\frac{(z-a)}{z-1/a}\Big|_{z=o}+\frac{-1}{a}\frac{(z-a)}{z}\Big|_{z=\frac{1}{a}}=\frac{-1}{a}$$

3) For $n<0$, $x(n)=\frac{1}{2\pi j}\oint_c\frac{-1}{a}\frac{z-a}{z-1/a}\frac{1}{z^{-(n-1)}}dz$

Therefore, it has a pole at zero with order of *(n–1)* and a pole at *1/a*. Since ROC$\equiv|z|>\frac{1}{|a|}\Rightarrow x(n)$ is right-sided and therefore, it is enough to find where it reaches the zero on the left side.

$$x(-1)=\frac{1}{2\pi j}\oint\frac{-1}{a}\frac{z-a}{z^2(z-1/a)}dz=\frac{-1}{a}\left[\frac{z-a}{z^2}\Big|_{z=\frac{1}{a}}+\frac{d}{dz}\left(\frac{z-a}{z-1/a}\Big|_{z=o}\right)\right]=0$$

With the same method, we can prove that *x(-2)=x(-3)=…=0* .

**The Inverse z-Transform by Power Series Expansion**

The method is to use long division. How to divide.

*Example:*

$$X(z)=\frac{1}{(1-z^{1})(1-2z^{-1})}=\frac{1}{1-3z^{-1}+2z^{-2}}=\frac{z^{2}}{z^{2}-3z+2}$$

It has two poles: $z = 1$ and $z = 2$. Therefore, if the signal is casual, then ROC: $|z| > 2$ and if the signal is non-casual, ROC: $|z| < 1$.

Case 1:

ROC: $|z| > 2 \rightarrow$ signal is casual and therefore $X(z)=\sum_{n=o}^{\infty} x(n)z^{-n}$ has terms with negative power for $z$. Therefore, we divide as:

$$\begin{array}{r|l}
 & 1+3z^{-1}+7z^{-2}+15z^{-3}+317^{-4}+\ldots \\
\hline
1-3z^{-1}+2z^{-2} & 1 \\
 & 1-3z^{-1}+2z^{-2} \\
 & \overline{0+3z^{-1}-2z^{2}} \\
 & 3z^{-1}-9z^{-2}+6z^{-3} \\
 & \overline{0\ 7z^{-2}-6z^{-3}} \\
 & 7z^{-2}-21z^{-3}+14z^{-4} \\
 & \overline{0+15z^{-3}-14z^{-4}} \\
 & 15z^{-3}-45z^{-4}+30z^{-5} \\
 & \overline{+32z^{-4}-30z^{-5}}
\end{array}$$

$$x(n) = \{\underset{\uparrow}{1}, 3, 7, 15, 31, \ldots\}$$

Case 2: ROC: $|z| < 1 \rightarrow$ signal is non-casual $\rightarrow$ $X(z)=\sum_{\infty}^{0} x(n)z^{-n}$ has terms with positive powers of $z$. Therefore, we divide in a way to get $z$ with positive powers.

$$
\begin{array}{r|l}
 & \frac{1}{2}z^{2}+\frac{3}{4}z^{3}+\frac{7}{8}z^{4}+\ldots \\
\hline
 & \quad 1 \\
 & \quad 1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{2} \\
\cline{2-2}
2z^{-2}-3z^{-1}+1 & 0+\frac{3}{2}z-\frac{1}{2}z^{2} \\
 & \quad \frac{3}{2}z-\frac{9}{4}z^{2}+\frac{3}{4}z^{3} \\
\cline{2-2}
 & \quad 0+\frac{7}{4}z^{2}-\frac{3}{4}z^{3}
\end{array}
$$

$$x(n)=\left\{\ldots,\frac{7}{8},\frac{3}{4},\frac{1}{2},0,\underset{\uparrow}{0}\right\}$$

Question: How would you use this method for a case like ROC: $1 < |z| < 2$?

**<u>The Inverse z-Transform by Partial- Fraction Expansion</u>**

$X(z)=\frac{N(z)}{D(z)}$ The goal is to write it in the form

$$X(z)=\frac{A_1}{\left(1-p_1z^{-1}\right)}+\frac{A_2}{\left(1-p_2z^{-1}\right)}+\frac{A_3}{\left(1-p_3z^{-1}\right)}+\ldots+\frac{A_N}{\left(1-p_Nz^{-1}\right)}$$

Let $X(z)=\frac{b_0+b_1z^{-1}+\ldots.+b_Mz^{-M}}{1+a_1z^{-1}+a_2z^{-2}+\ldots+a_Nz^{-N}}$. If it is an improper rational function *($M \geq N$)* then the first step is to write it as:

$X(z)=c_0+c_1z^{-1}+\ldots..+c_{M-N}z^{-(M-N)}+\frac{N_1(z)}{D(z)}$, where now the degree of $N_1(z)$ is less than $N$.

The inverse z-transformation of $c_0 + c_1z^{-1} + \ldots.$ is very straight-forward. Next step is to write $\frac{N_1(z)}{D(z)}$ or in general a proper function $\frac{N(z)}{D(z)}$, where $M < N$, in terms of positive powers of $z$, factorize denominator and then write $\frac{X(z)}{z}$ in terms of partial fractions. In general, lets assume an $X(z)$ has $N$ simple poles and L multiple poles at $k = j$. Then:

$$\frac{X(z)}{z}=\frac{A_1}{z-p_1}+\frac{A_2}{z-p_2}+\ldots.+\frac{A_{j1}}{1-p_j}+\frac{A_{j2}}{\left(1-p_j\right)^2}+\ldots.+\frac{A_{j\ell}}{\left(1-p_j\right)^{\ell}}+\ldots\frac{A_N}{\left(1-P_N\right)}$$

Then $A_k = \frac{(z-p_k)X(z)}{z}\Big|_{z=p_k}$ $k = 1, 2, ...., n$ excluding $k = j$ and

$$A_{jk} = \frac{d^{\ell-k}}{d^{\ell-k}z}\left[\frac{(z-p_i)^k X(z)}{z}\right], k = 1, 2, .....,l$$

Then

$$Z^{-1}\left\{\frac{1}{1-p_k z^{-1}}\right\} = \begin{cases} (p_k)^n u(n) & ROC: |z| > |p_k| \\ -(p_k)^N u(-n-1) & ROC: |z| < |p_k| \end{cases}$$

**Example**

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}} = \frac{z^2}{(z-1)\left(z-\frac{1}{2}\right)}$$

$$\frac{X(z)}{z} = \frac{z}{(z-1)\left(z-\frac{1}{2}\right)} = \frac{A_1}{z-1} + \frac{A_2}{z-\frac{1}{2}}$$

$$A_1 = (z-1)\frac{X(z)}{z}\Big|_{z=1} = \frac{1}{\frac{1}{2}} = 2$$

$$A_2 = \left(z-\frac{1}{2}\right)\frac{X(z)}{z}\Big|_{z=\frac{1}{2}} = \frac{\frac{1}{2}}{-\frac{1}{2}} = -1$$

$$\rightarrow \qquad X(z) = z\left[\frac{2}{z-1} - \frac{1}{z-\frac{1}{2}}\right] = \frac{2}{1-z^{-1}} - \frac{1}{1-\frac{1}{2}z^{-1}}$$

Now depending on ROC, we get different *x(n)*.

1) ROC: $|z| > 1$ casual

$$\rightarrow x(n) = 2(1)^n u(n) - \left(\tfrac{1}{2}\right)^n u(n) = \left(2-0.5^n\right)u(n)$$

2) ROC: $|z| < ½$ non-casual and left-sided

$$\begin{aligned} x(n) &= -2(1)^n u(-n-1) + 0.5^n u(-n-1) \\ &= \left(-2+0.5^n\right)u(-n-1) \end{aligned}$$

3) $\frac{1}{2} < |z| < 1$ non-casual and two-sided

$$x(n) = -2(1)^n u(-n-1) - (0.5)^n u(n)$$

**One-Sided Z Transform**

One-sided Z Transform is defined as $X^+(z) = \sum_{n=o}^{\infty} x(n) z^{-n} = \sum_{-\infty}^{+\infty} x(n) u(n) z^{-n}$. It does not contain information about *x(n)* for *n < 0*. It is unique only for the causal signals that are zero for *n < 0*. It is useful to solve difference equations of the systems that are not relaxed initially, but the input is not necessarily zero before applying to the system.

Properties

If $x(n) \xleftrightarrow{Z^+} X^+(z)$ then $x(n-k) \xleftrightarrow{Z^+} z^{-k}\left[X^+(z) + \sum_{n=1}^{k} x(-n) z^n\right], k > o$

Proof: $Z^+\{x(n-k)\} = \sum_{n=o}^{\infty} x(n-k) z^{-n}$. Let *n – k = l* then *n = 0* $\rightarrow$ *l = -k* and also *-n = -l –k.*

$$= \sum_{\ell=-K}^{\infty} x(\ell) z^{-\ell} z^{-K}$$

$$= z^{-K}\left[\sum_{\ell=-K}^{-1} x\left(\underset{-n}{\underset{\Downarrow}{\ell}}\right) z^{-\ell} + \underbrace{\sum_{\ell=o}^{\infty} x(\ell) z^{-\ell}}_{x^+(z)}\right] = z^{-K}\left[\sum_{n=1}^{K} x(-n) z^n + X^+(z)\right]$$

Time Advance: $x(n+k) \xleftrightarrow{Z^+} z^k\left[X^+(z) - \sum_{n=o}^{k-1} x(n) z^{-n}\right], k > 0$

Final Value Theorem: $\lim_{n\to\infty} x(n) = \lim_{z\to 1} (z-1)\, X^+(z)$

This limit exists if the ROC of $(z-1)X^+(z)$ includes the unit circle.

This can be proved by the following analogy. If the limit $\lim_{n\to\infty} x(n)$ exists, then the function x(n) can be written as $x(n) = c + f(n)$, where $\lim_{n\to\infty} x(n) = c$ and $\lim_{n\to\infty} f(n) = 0$. Then take the one sided Z-Transform from both sides and prove the theorem.

**Analysis of the Systems in Z-Domain**

Without too much restriction, lets assume

$$x(z) = \frac{N(z)}{Q(z)} \text{ and } H(z) = \frac{B(z)}{A(z)}$$

$\rightarrow Y(z)=\dfrac{B(z)\cdot N(z)}{Q(z)\cdot A(z)}$. If the system is relaxed, the initial condition = $o$

If the system is not relaxed, then

$$Y(z)=\underbrace{H(z)\cdot x(z)}_{Y_{zs}(z)}+\underbrace{\frac{N_o(z)}{A(z)}}_{Y_{zi}^{+}(z)} \quad \text{where } N_0(z)=-\sum_{k=1}^{N}a_k z^{-k}\cdot\sum_{n=1}^{k}y(-n)z^{-n}$$

$H(z)$ has $p_1, \ldots, p_N$ poles and $X(z)$ has $q_1, \ldots, q_L$ poles.

First lets assume that the poles are distinct and not common. Then

$$Y(z)=\sum_{k=1}^{N}\frac{A_k}{1-P_k z^{-k}}+\sum_{k=1}^{L}\frac{Q_k}{1-q_k z^{-k}}$$

$$y(n)=\underbrace{\sum_{k=1}^{N}A_k(P_k)^n u(n)}_{natural\ response}+\underbrace{\sum_{k=1}^{L}Q_k(q_k)^n u(n)}_{force\ response}$$

Note that natural response ≠ zero-input response. It is in fact the no-input response.

**Example**

$y(n)=\frac{1}{2}y(n-1)+x(n)$ Find the output when $x(n)=10\cos\frac{\pi n}{4}u(n)$

**Solution**

$$Y(z)=\frac{1}{2}z^{-1}Y(z)+X(z)$$

$$\rightarrow H(z)=\frac{1}{1-\frac{1}{2}z^{-1}} \qquad P_1=\frac{1}{2}$$

However, $X(z)=\dfrac{10\left(1-\frac{1}{\sqrt{2}}z^{-1}\right)}{1-\sqrt{2}z^{-1}+z^{-2}} \Rightarrow q_1=e^{j\pi/4} \qquad q_2=q_1^{*}=e^{-j\pi/4}$

$$Y(z)=H(z)\cdot x(z)=\underbrace{\frac{6.3}{1-1/2z^{-1}}}_{natrual\ response}+\underbrace{\frac{6.78e^{-j28.7^o}}{1-e^{j\pi/4}z^{-1}}+\frac{6.78e^{j28.7^o}}{1-e^{-j\pi/4}z^{-1}}}_{force\ response}$$

$$y_{nr}(n)=6.3\left(\frac{1}{2}\right)^n u(n)$$

$$y_{fr}(n)=13.5\cos\left(\frac{\pi}{4}n-28.7^o\right)u(n)$$

**Testing it with MATLAB:**

$$Y(z)=\frac{10\left(1-\frac{1}{\sqrt{2}}z^{-1}\right)}{1-\left(\sqrt{2}+\frac{1}{2}\right)z^{-1}+\left(1-\frac{\sqrt{2}}{2}\right)z^{-2}-\frac{1}{2}z^{-3}}$$

$$b=\left[10,-10/\sqrt{2}\right];\ a=\left[1,-\left(\sqrt{2}+\frac{1}{2}\right),\left(1+\frac{\sqrt{2}}{2}\right),-\frac{1}{2}\right];$$

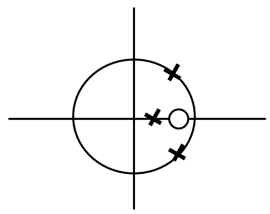
*[R, P, K] = residuez (b, a)*

$$R=\begin{bmatrix}5.9537-3.2562i\\5.9537+3.2562i\\-1.9074\end{bmatrix}\qquad P=poles\begin{bmatrix}0.7071+0.7071i\\0.7071-0.7071i\\0.5\end{bmatrix}K=[\ ]$$

norm (R (1)) = 6.78

angle (R (1)) = -0.5005 rad = - 28.7$^{o}$

zplane (b, a) % plots the zero-poles



**Example of a Non-Relaxed System**

$$y(n)-\frac{3}{2}y(n-1)+\frac{1}{2}y(n-2)=x(n)\qquad n\geq 0$$

$x(n)=\left(\frac{1}{4}\right)^{n}u(n)$ find y(n) if *y (-1) = 4* and *y (-2) = 10*

***Solution***

Taking $Z^{+}$ from both sides:

$$Y^{+}(z)-\frac{3}{2}\left[y(-1)+z^{-1}Y^{+}(z)\right]+\frac{1}{2}\left[y(-2)+z^{-1}y(-1)+z^{-2}Y^{+}(z)\right]=X(z)=\frac{1}{1-\frac{1}{4}z^{-1}}$$

$$Y^{+}(z)\left[1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{2}\right]=\frac{1}{1-\frac{1}{4}z^{-1}}+\left(1-2z^{-1}\right)$$

Finally: $Y^{+}(z)=\dfrac{2-\frac{9}{4}z^{-1}+\frac{1}{2}z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$

or $Y^{+}(z)=\dfrac{1}{1-\frac{1}{2}z^{-1}}+\dfrac{\frac{2}{3}}{1-z^{-1}}+\dfrac{\frac{1}{3}}{1-\frac{1}{4}z^{-1}}$

$$y(n)=\left[\underbrace{\left(\frac{1}{2}\right)^n+\frac{2}{3}}_{natural\ response}+\frac{1}{3}\underbrace{\left(\frac{1}{4}\right)^n}_{force\ response}\right]u(n)$$

Note that *natural response* is due to system poles and *force response* is due to the input poles. *Transient response* is due to the poles inside the unit circle and *steady-state response* is due to poles on the unit circle. In this case,

$$Transient\ Response \equiv \left[\frac{1}{3}\left(\frac{1}{4}\right)^n+\left(\frac{1}{2}\right)^n\right]u(n)$$

and *Steady-State Response* $\equiv \frac{2}{3}u(n)$.

*Zero-Input (or Initial condition) Response* is: $Y_{ZI}(z)=H(z).X_{IC}(z)$ and *Zero-State Response* is: $Y_{ZS}(z)=H(z).X(z)$. In this case,

$$y_{zs}(n)=\left[\frac{1}{3}\left(\frac{1}{4}\right)^n-2\left(\frac{1}{2}\right)^n+\frac{8}{3}\right]u(n)$$

and $y_{zi}(n)=\left[3\left(\frac{1}{2}\right)^n-2\right]u(n)$.

Note that complete response is either *Transient Response+Steady-State Response*, or *Natural Response + Force Response*, or *Zero-Input Response + Zero-State Response*. Each response emphasizes a different aspect of system analysis.

Checking with MatLab:

*n = [0: 7]*; % just checking the first 8 samples

$x=\left(\frac{1}{4}\right)\hat{}\ n;$ *xic = [1, -2]* % terms due to initial conditions *(1 – 2z⁻¹)*

*b = [1, 0];*

$a=\left[1-\frac{3}{2}\ \frac{1}{2}\right];$

$y_1$ *= filter (b, a, x, xic)*;

$y=\left(\frac{1}{3}\right)*\left(\frac{1}{4}\right)\hat{}\ n+\frac{1}{2}\hat{}\ n+\frac{2}{3}*ones(1,8);$

*Since y* and $y_1$ are the same, then our solution is correct!

However, for large order difference equations, it is tedious to determine *xic(n)* analytically. MatLab command filtic does find *xic* as well.

*Xic = filtic (b, a, Y, x*) where *Y* and *x* are initial conditions.

$$Y = [y(-1), y(-2), \ldots, y(-N)]$$

$$x = [x(-1), x(-2), \ldots, x(-M)]$$

If $x(n) = 0$ for $n \leq -1$, then x need not to be defined. In our example:

$$Y = [4, 10];$$

$$Xic = 1, -2$$

**Causality and Stability**

If $h(n) = 0$, for $n < 0$, then the system is causal. Then its ROC is the exterior of a circle. The stability of a system is quarantined by the condition that the ROC includes the unit circle. Because the necessary and sufficient condition for a BIBO system is that $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$. It follows that $|H(z)| = \left|\sum_{n=-\infty}^{\infty} h(n) z^{-n}\right| \leq \sum_{n=-\infty}^{\infty} \left|h(n) z^{-n}\right| = \sum_{n=-\infty}^{\infty} |h(n)| \left|z^{-n}\right|$. When evaluated on the unit circle, $|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)|$. Note that causality and stability are independent of each other. One doesn't imply the other. However, a causal LTI system is BIBO if and only if all the poles of *H(z)* are inside the unit circle.

**Pole-Zero Cancellation**

Example: *y(n) = 2.5 y(n - 1) – y(n - 2) +x(n) – 5x(n – 1) + 6 x(n – 2)*

$$H(z) = \frac{1 - 5z^{-1} + 6z^{-2}}{1 - 2.5z^{-1} + z^{-2}}$$

$$H(z) = \frac{\left(1 - 2z^{-1}\right)\left(1 - 3z^{-1}\right)}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 - 2z^{-1}\right)} \qquad \begin{array}{l} P_1 = \frac{1}{2}, P_2 = 2 \\ z_1 = 3, z_2 = 2 \end{array}$$

It seems that the pole at 2 is cancelled by zero at 2. So, the system is theoretically stable but not practically.

*Do problems: 3,6,7,9,15,22 and 43 of Chapter 3.*