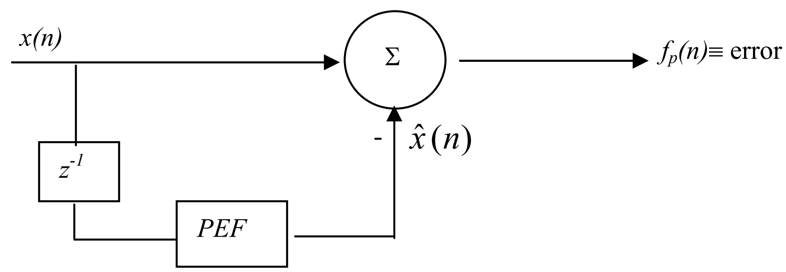
Forward Linear Prediction

If we assume an AR model for a stochastic process, it means that we can predict the future values from a limited observations of its past values.

(*1) $\hat{x}(n) = -\sum_{k=1}^{P} a_p(k) x(n-k)$, P is the order of the system (filter) where $\{-a_p(k)\}$ are the tap-weights and are called Prediction Coefficients of one-step forward linear prediction error filter (PEF).



The prediction error: $f_p(n) = x(n) - \hat{x}(n) \Rightarrow f_p(n) = x(n) + \sum_{k=1}^{P} a_p(k) x(n-k)$ using Eq. (*1).

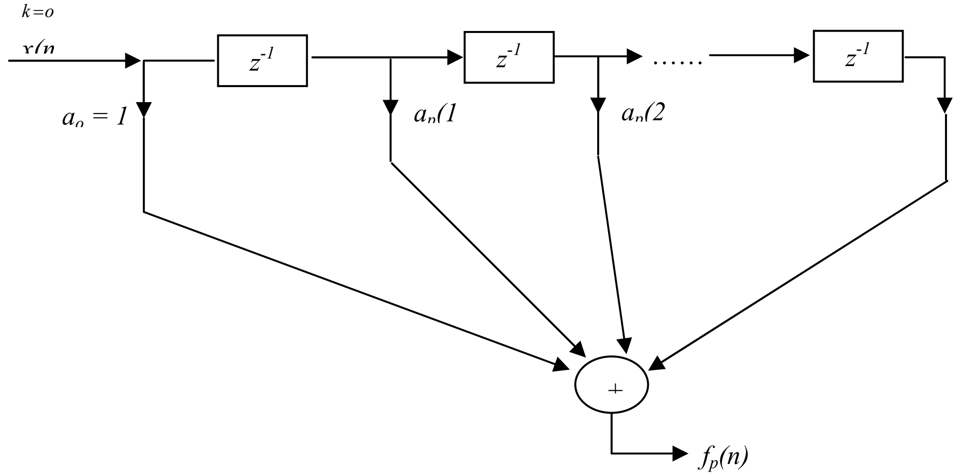
$f_p(n) = \sum_{k=o}^{P} a_p(k) x(n-k)$ with $a_o = 1$ (*2). Comparing this equation with AR model leads to: $f_p(n) = w(n)$.

Take Z-Transform from both sides of Eq. (*2):

$$F_p(z) = A_p(z) \cdot X(z) \Rightarrow A_p(z) = \frac{F_p(z)}{X(z)} = \frac{F_p(z)}{F_o(z)}$$

Note that $X(z)$ is in fact, $F_o(z)$ as the Eq. (*2) is a recursive equation in nature.

$A_p(z) = \sum_{k=o}^{P} a_p(k) z^{-k}$ with $a_o = 1$. This is a FIR filter (all-zeros).



How to find the filter coefficients, $a_p(k)$?

A way to solve is to minimize the variance of the error $f_p(n)$. That is:

$$\varepsilon_p^f = E\left\{\left|f_p(n)\right|^2\right\} = E\left\{\left[x(n)+\sum_{k=1}^{P}a_p(k)x(n-k)\right]\left[x(n)+\sum_{\ell=1}^{P}a_p^*(\ell)x(n-\ell)\right]\right\}$$

$$\Rightarrow\ \varepsilon_p^f = \gamma_{xx}(0)+2\,\mathrm{Re}\left\{\sum_{k=1}^{P}a_p(k)\gamma_{xx}(k)\right\}+\sum_{k=1}^{P}\sum_{\ell=1}^{P}a_p^*(\ell)a_p(k)\gamma_{xx}(\ell-k)$$

This is a quadratic function of the tap-weights $\{a_p(k)\}$ and it has a ball-shaped $(P+1)$ dimensional surface. This surface has a unique minimum. At the minimum point, the gradient vector $\nabla\varepsilon_k^f = 0$ for $k=0,1,\ldots,P-1$ independently. If we let

$a_p(k)=\alpha_k + j\beta_k$ in general, then $\nabla\varepsilon_k^f = \dfrac{\partial\varepsilon_k^f}{\partial\alpha}+j\dfrac{\partial\varepsilon_k^f}{\partial\beta}$.

Taking this gradient vector and making it equal to zero, leads to the equation:

$$\boxed{\gamma_{xx}(\ell) = -\sum_{k=1}^{P}a_p(k)\gamma_{xx}(\ell-k),\ \ell = 1,2,\ldots\ldots\ldots\ldots,p}$$

This equation is called ***"Normal Equation"***. In matrix form:

$$\sum_{k=o}^{P}a_p(k)\gamma_{xx}(\ell-k)=0 \quad \ell=1,2,\ldots p \ \text{ and } \ a_p(0)=1$$

$$\boxed{\underline{\Gamma_{xx}}(n)\cdot\underline{a_p} = \underline{0}}$$

With this solution, the minimum mean-square prediction error will be:

$$\boxed{\min\left[\varepsilon_p^f\right] = E_p^f = \gamma_{xx}(0)+\sum_{k=1}^{P}a_p(k)\gamma_{xx}(-k)}$$

Backward Linear Prediction

One-step backward predictor of order p:

$$\hat{x}(n-p) = -\sum_{k=o}^{p-1}b_p(k)x(n-k)$$

Backward Prediction error: $g_p(n) = x(n-p)-\hat{x}(n-p)$

$$\rightarrow g_p(n) = x(n-p) + \sum_{k=0}^{p-1} b_p(k)x(n-k) = \sum_{k=0}^{p} b_p(k)x(n-k) \text{ where } b_p(p) = 1$$

Therefore, backward linear prediction filter can be realized either by a direct-form FIR filter structure similar to forward linear prediction filter or as a lattice structure. Note that: $b_p(k) = a_p^*(p-k)$.

Also, we can write: $G_p(z) = B_p(z) \cdot X(z)$

$$B_p(z) = \frac{G_p(z)}{X(z)} = \frac{G_p(z)}{G_o(z)}$$

Also that $B_p(z) = \sum_{k=0}^{p} b_p(k) z^{-k}$

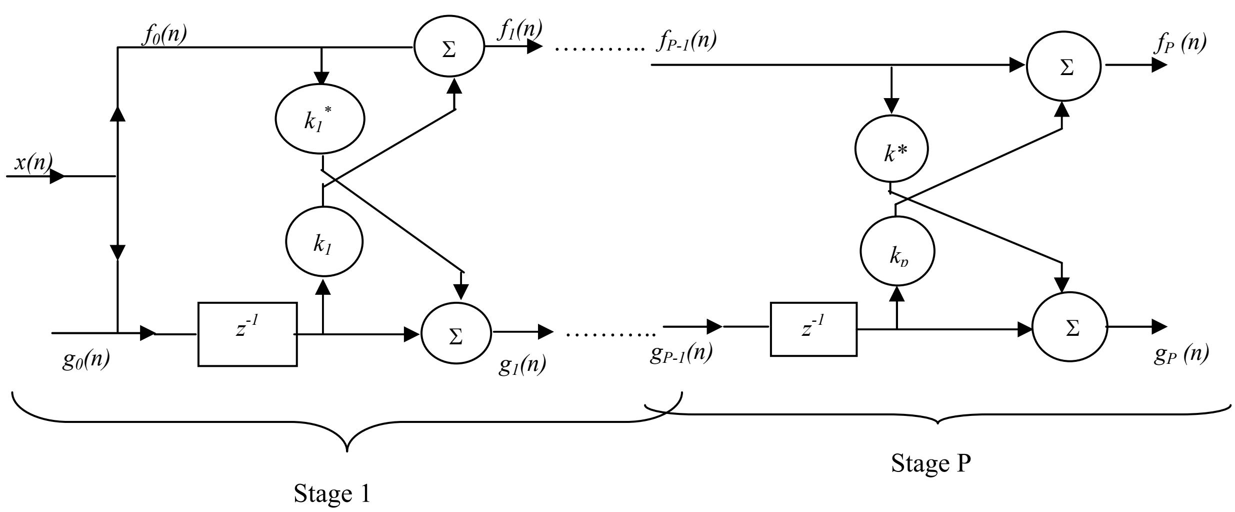
$$= \sum_{k=0}^{p} a_p^*(p-k) z^{-k} \quad let\ p-k = k' \Rightarrow -k = k' - p$$

$$= \sum_{k'=p}^{0} a_p^*(k') z^{k'-p} = z^{-p} \overbrace{\sum_{k=0} a_p^*(k) z^{k}}^{k' = k} = z^{-p} A_p^*(z^{-1})$$

$$\boxed{\therefore B_p(z) = z^{-p} A_p^*(z^{-1})}$$

This implies that the zeros of the FIR filter with system function $B_p(z)$ are simply the conjugate reciprocals of the zeros of $A_p(z)$. Hence, $B_p(z)$ is called the reciprocal or reverse polynomial of $A_p(z)$.

FIR Lattice Structure



$$(*)\begin{cases} f_0(n) = g_0(n) = x(n) & \\ f_m(n) = f_{m-1}(n) + k_m g_{m-1}(n-1) & m = 1, 2, \ldots\ldots, p \\ g_m(n) = k_m^* f_{m-1}(n) + g_{m-1}(n-1) & \end{cases}$$

k_m are called to reflection coefficients. Note: $k_m = a_p(p)$

In order to derive $a_p(k)$ from k_m, take the Z-transform of Equations (*)

$$\begin{cases} F_0(z) = G_0(z) = X(z) & \\ F_m(z) = F_{m-1}(z) + k_m z^{-1} G_{m-1}(z) & m = 1, 2, \ldots\ldots, p \\ G_m(z) = k_m^* F_{m-1}(z) + z^{-1} G_{m-1}(z) & \end{cases}$$

Now replace $F_m(z) = A_m(z)\cdot X(z)$ and $G_m(z) = B_m(z)\cdot X(z)$ and cancel $X(z)$ from both sides.

Then we get:

$$\begin{cases} A_o(z) = B_o(z) = 1 \\ A_m(z) = A_{m-1}(z) + k_m z^{-1} B_{m-1}(z) \\ B_m(z) = k_m^* A_{m-1}(z) + z^{-1} B_{m-1}(z) \end{cases}$$

or $\begin{bmatrix} A_m(z) \\ B_m(z) \end{bmatrix} = \begin{bmatrix} 1 & k_m z^{-1} \\ k_m^* & z^{-1} \end{bmatrix} \begin{bmatrix} A_{m-1}(z) \\ B_{m-1}(z) \end{bmatrix}$

$$\Rightarrow \boxed{A_{m-1}(z) = \frac{A_m(z) - k_m B_m(z)}{1 - |k_m|^2}}$$

Recalling that $a_m(0) = 1$ and $a_m(m) = k_m$, we can also write:

$$\boxed{a_{m-1}(k) = \frac{a_m(k) - \overbrace{a_m(m)}^{k_m} \cdot \overbrace{a_m^*(m-k)}^{b_m(k)}}{1 - |a_m(m)|^2}}$$

The point is that a direct FIR structure to derive $a_p(k)$ requires $\frac{p(p+1)}{2}$ filter coefficients (due to stages $A_1(z), A_2(z), \ldots . A_p(z)$, while the lattice structure needs only p, $\{k_1, k_2, \ldots, k_p\}$, coefficients.

Also: $\varepsilon_p^b = E\left\{|g_p(n)|^2\right\}$ and $\min[\varepsilon_p^b] = E_p^b = E_p^f$ and $|k_m| \le 1$. If $|k_m| = 1$, the recursive equations breaks down. $|k_m| = 1$ is indicative that $A_{m-1}(z)$ has roots on the unit circle. Also note that: $\boxed{E_m^f = \left(1 - |k_m|^2\right) E_{m-1}^f}$, which is a monotonically decreasing sequence.

Relationship Between AR Process and Linear Prediction Error Filter ***(important)***

If a process *x(n)* is really an AR process, then $a_p(k)$, the coefficients of the Prediction Error Filter (PEF), are in fact, the same as AR parameters in Yull-Walker equation and minimum MSE at the p^{th} order is in fact σ_w^2 and therefore, the PEF has become optimized.

If *x(n)* is not an AR process, still the PEF coefficients are the best approximates of the AR parameters that can represent *x(n)*.

Example

Consider the following AR process:

$x(n)+c_1x(n-1)+c_2x(n-2)=w(n)$ where $c_1 = -0.1$ and $c_2 = -0.8$ and $\sigma_w^2 = 0.27$

a) Find σ_x^2

b) Find the reflection coefficients (k_m)

c) Find min mean-squared error E_m

Solution

Note that $a_2(0)=1$, $a_2(1)= c_1=-0.1$ and $a_2(2)= c_2=-0.8$

a) $\sigma_x^2 = \gamma_{xx}(0)$

Using the Yull-Walker equation, we have:

$$\begin{bmatrix} \gamma(0) & \gamma(1) & \gamma(2) \\ \gamma(1) & \gamma(0) & \gamma(1) \\ \gamma(2) & \gamma(1) & \gamma(0) \end{bmatrix} \begin{bmatrix} 1 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \sigma_w^2 \\ 0 \\ 0 \end{bmatrix}$$

To solve for $\gamma(o), \gamma(1)$ and $\gamma(2)$, rewrite it as:

$$\underbrace{\begin{bmatrix} 1 & c_1 & c_2 \\ c_1 & 1+c_2 & 0 \\ c_2 & c_1 & 1 \end{bmatrix}}_{\substack{\text{Matrix A} \\ \Delta = |\underline{A}|}} \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \end{bmatrix} = \begin{bmatrix} \sigma_w^2 \\ 0 \\ 0 \end{bmatrix}$$

Using Cramer rule: $$\gamma(0) = \frac{\begin{vmatrix} \sigma_w^2 & c_1 & c_2 \\ 0 & 1+c_2 & 0 \\ 0 & c_1 & 1 \end{vmatrix} = \sigma_w^2(1+c_2)}{\Delta}$$

$$\gamma(0)=\frac{(1+c_2)\sigma_w^{\ 2}}{(1-c_2)\left((1+c_2)^2-c_1^2\right)}=1 \text{ for this example.}$$

b) $k_2 = a_2(2)=c_2 =-0.8$

Using recursive equation: $a_2(1)=-0.1=a_1(1)+k_2a_1(1)\Rightarrow$ $-0.1= a_1(1).(1-0.8) \Rightarrow$

$a_1(1)= k_1 =-\dfrac{1}{2}$

c) $E_0=\sigma_x^2=\gamma(0)=1$

$$E_1=E_0\left(1-|k_1|^2\right)=1-\frac{1}{4}=\frac{3}{4}$$

$$E_2=E_1\left(1-|k_2|^2\right)=\frac{3}{4}\left(1-\frac{64}{100}\right)=\frac{3}{4}\times\frac{36}{100}=0.27=\sigma_w^{\ 2}$$

More Examples

Determine the lattice coefficients corresponding to the FIR filter with system function

$$H(z)=A_2(z)=1+\frac{3}{8}z^{-1}+\frac{1}{2}z^{-2}$$

Solution

$$p=2 \text{ and } a_p=\left[\underset{a_2(0)}{1},\underset{a_2(1)}{\frac{3}{8}},\underset{a_2(2)}{\frac{1}{2}}\right]\Rightarrow k_2=a_2(2)=\frac{1}{2}$$

$$B_2(z)=z^{-2}A_2\left(z^{-1}\right)=z^{-2}\left[1+\frac{3}{8}z+\frac{1}{2}z^2\right]=\frac{1}{2}+\frac{3}{8}z^{-1}+z^{-2}$$

$$A_1(z)=\frac{A_2(z)-k_2B_2(z)}{1-|k_2|^2}=\frac{1}{1-\frac{1}{4}}\left[1+\frac{3}{8}z^{-1}+\frac{1}{2}z^{-2}-\frac{1}{2}\left(\frac{1}{2}+\frac{3}{8}z^{-1}+z^{-2}\right)\right]$$

$$=\frac{4}{3}\left[1+\frac{3}{16}z^{-1}-\frac{1}{4}\right]=1+\underset{a_1(1)}{\frac{1}{4}}z^{-1}$$

$$\Rightarrow k_1=a_1(1)=\frac{1}{4}$$

<u>Example P. 11.7</u>

Determine the impulse response of FIR filter described by lattice coefficients $k_1 = 0.6, k_2 = 0.3, k_3 = 0.5, k_4 = 0.9$

<u>Solution</u>

$A_0(z) = B_0(z) = 1$

$$\begin{cases} A_1(z) = A_0(z) + k_1 z^{-1} B_0(z) = 1 + 0.6z^{-1} \\ B_1(z) = k_1 A_0(z) + z^{-1} B_0(z) = 0.6 + z^{-1} \end{cases}$$

$$\begin{cases} A_2(z) = A_1(z) + k_2 z^{-1} B_1(z) = 1 + 0.6z^{-1} + 0.3z^{-1}\left(0.6 + z^{-1}\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad = 1 + 0.78z^{-1} + 0.3z^{-2} \\ B_2(z) = k_2 A_2(z) + z^{-1} B_1(z) = 0.3 + 0.78z^{-1} + z^{-2} \end{cases}$$

with the same routine.

$$\begin{cases} A_3(z) = 1 + 0.93z^{-1} + 0.69z^{-2} + 0.5z^{-3} \\ B_3(z) = 0.5 + 0.69z^{-1} + 0.93z^{-2} + z \end{cases}$$

Finally, $A_4(z) = H(z) = 1 + 1.38z^{-1} + 1.311z^{-2} + 1.337z^{-3} + 0.9z^{-4}$. If it was asked to determine an all-pole filter corresponding to the same lattice coefficients, then $H(z)$ would have been $\frac{1}{A_4(z)}$. See problem 11.22.