DIGITAL SIGNAL PROCESSING

Lecture 1 - Chapter 1

Classification of Signals: Continuous-Time verses Discrete-Time Signals

Continuous time or analog signals are signals that are defined for every value of $a < t < b$, where *(a, b)* can be $(-\infty, +\infty)$, i.e., $x\,(t) = e^{-|t|}$ or $x(t) = \cos(pt)$. .

Discrete-time signals are defined at discrete-time instants and between the two discrete time instants are undefined but are not zero. They can be obtained either by sampling analog signals or they can be discrete in nature like discrete measurement signals.

A discrete-time signal having a set of discrete values is called a digital signal. Note that sampling an analog signal produces a discrete-time signal. Then quantization of its values produces a digital signal.

Deterministic versus Random Signals

Any signal that can be uniquely described by an explicit mathematical expression or a well-defined rule is called “deterministic”. The past, present and future of a deterministic signal are known with certainty. Otherwise, it is called “Random” and its properties is explained by statistical techniques.

Review of Sinusoids in Continuous and Discrete Time

$$\begin{aligned} x_a(t) &= A\cos(Wt + q) \qquad -\infty < t < \infty \\ &= A\cos(2pFt + q) \quad W = 2pF \end{aligned}$$

$x_a(t + T_p) = x_a(t),\;\; T_p = \dfrac{1}{F}$: fundamental period. Increasing F means increasing oscillation in time domain. $F = 0$ corresponds to $Tp = \infty$.

Also, for complex exponential signals, $x_a(t) = Ae^{j(Wt+q)}$. A sinusoidal signal then can also be expressed as

$$x_a(t) = A\cos(Wt+q) = \frac{1}{2}Ae^{j(Wt+q)} + \frac{1}{2}Ae^{-j(Wt+q)}$$

Discrete-Time Sinusoid Signals

$$x(n) = A\cos(w_0 n + q) \quad w = 2p\ f$$
$$x(n) = A\cos(2p\ f_0 n + q) \quad -\infty < n < \infty$$

A few important differences between continuous sinusoid and discrete sinusoids:

1) A discrete-time sinusoid is periodic only if its frequency is a rational number.

By default, $x(n+N) = x(n)$ for all n if *x(n)* is periodic. The smallest N is called Fundamental Period.

$$x(n+N) = A\cos(2p\ f_0(n+N)+q) = \cos(2p\ f_0 n + q)$$

This relationship is true if and only if $2pf_0 N = 2Kp$

$\Rightarrow f_0 = \dfrac{K}{N}$: a rational number.

To determine the period N of a periodic discrete time sinusoid, we express f as two relatively prime numbers. Observe that a small change in frequency can result in a large change in period. For example, $x_1(n) = A\cos\left(2p\left(\frac{1}{2}\right)n + q\right) = A\cos(p\ n + q)$ its period is $N_1 = \dfrac{K}{f_1} = 2 = \dfrac{30}{60}$.

Now consider $x_2(n) = A\cos\left(2p\left(\frac{31}{60}\right)n + q\right)$, $f_2 = \dfrac{31}{60} = 0.517 \rightarrow \underline{N_2 = 60}$

2. An analog $F(-\infty, +\infty)$ maps to $-\dfrac{1}{2} \le f \le \dfrac{1}{2}$ or equivalently to $-p \le w \le p$ or in other words, the highest rate of oscillation occurs at $f = \pm\dfrac{1}{2}$ or $w = \pm p$. To see what happens for $p \le w \le 2p$ consider $w_1 = w_0$ and $w_2 = 2p - w_0$. When w_1 varies between p to $2p$, then w_2 varies between p and 0. Now

$$x_1(n) = A\cos w_1 n = A\cos w_0 n$$

$$x_2(n) = A\cos(w_2)n = A\cos(2p - w_0)n =^{*} A\cos(-w_o n) = A\cos w_o n = x_1(n)$$

* this is only true because *n* is an integer value, i.e., *x(n)* is a discrete signal.

Hence, $x_2(n) = x_1(n)$ is an alias because $-\infty < F < +\infty$ maps only to $-\frac{1}{2} \le f \le \frac{1}{2}$ by sampling.

Analog to Digital Conversion (A/D)

- sampling (sampling rate)
- quantization

Sampling:
$$\begin{aligned} x(n) &= x_a(nT_s), \quad \text{where} -\infty < n < \infty \\ &= x_a(t)\big|_{t=nTs} \end{aligned}$$

$t = nT_s = \frac{n}{F_s}$, F_s = Sampling rate (Frequency) (Hz)

Consider an analog sinusoid: $x_a(t) = A\cos(2p\ Ft + f)$

$$x(n) = x_a(nT_s) = A\cos(2p\ F\, nT_s + f) = A\cos\left(2p\ \frac{F}{F_s}n + f\right)$$

Now recall that $-\infty < F < \infty$ maps to $-\frac{1}{2} \le f \le \frac{1}{2}$

$$\rightarrow \frac{-1}{2} \le f = \frac{F}{F_s} \le \frac{1}{2} \quad \rightarrow \quad \boxed{F_s \ge 2F_{\max}} \quad \text{Nyquist rate}$$

Example 1:
$$\begin{aligned} x_{a1}(t) &= \cos 20p\ t \quad \Rightarrow F_1 = 10Hz \\ x_{a2}(t) &= \cos 100p\ t \quad \Rightarrow F_2 = 50Hz \end{aligned}$$

If we digitize both of these signals with Fs = 40Hz, then

$$x_1(n) = x_{a1}(nT_s) = x_{a1}\left(\frac{n}{40}\right) = \cos\frac{20p}{40}n = \cos\frac{p}{2}n$$

$$x_2(n) = x_{a2}\left(\frac{n}{40}\right) = \cos\frac{100p}{40}n = \cos\frac{5p}{2}n = \cos\left(2p + \frac{p}{2}\right)n$$

$$= \cos\frac{p}{2}n = x_1(n)!!$$

Therefore, by this sampling rate $x_2(n)$ has become same as $x_1(n)$, which is an aliasing error. Equivalently, in this case, the frequency of 50 Hz is an alias of 10 Hz by sampling rate of 40 Hz. Furthermore, all frequencies (F_1 + 40K) are aliases of F_1. Hence, do not use the Nyquist rate blindly.

Example 2: $x_a(t) = \underbrace{3\cos 50p\ t}_{x_1} + \underbrace{10\sin 300p\ t}_{x_2} - \underbrace{\cos 100p\ t}_{x_3}$

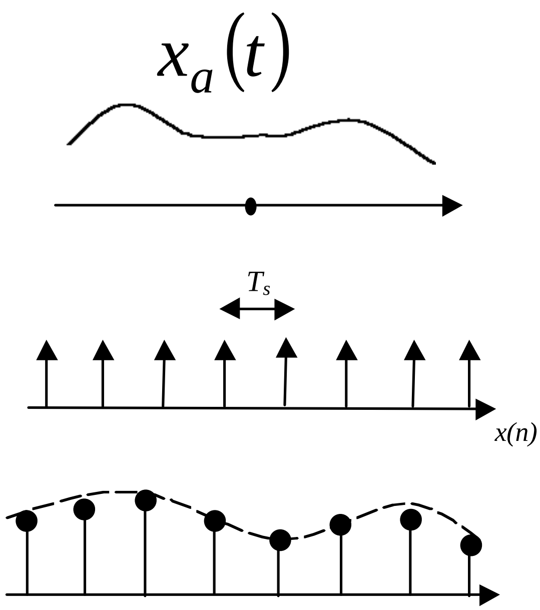
Nyquist rate? $f_1 = 25, f_2 = 150, f_3 = 50\ Hz$

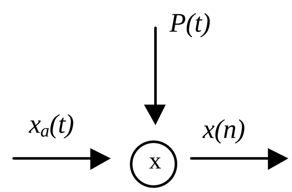
$\rightarrow F_{max} = 150\ Hz$ ® $F_s = 300\ Hz?!$ ◄

Problem: with $F_x = 300\ Hz$, $x_2(n) = x_{a2}\left(\frac{n}{F_s}\right) = 10\sin p\ n = 0$ all the time!

If it had a phase shift $o < q < n$, then it would have been fine, but it is best to choose a higher sampling rate.

Sampling





$$x_p(t) = x_a(t) \cdot p(t)$$

$$p(t) = \sum_{-\infty}^{+\infty} d\,(t - nT_s)$$

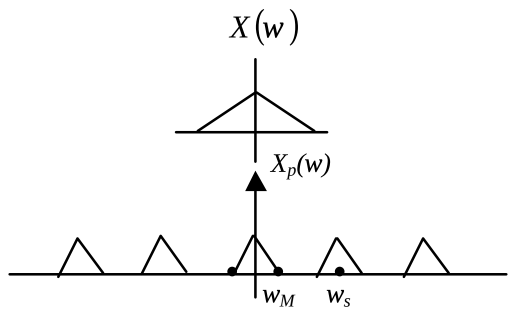
$$x(n) = x_p(t)|_{t=nT_x} = \sum_{-\infty}^{+\infty} x(nT_s)d\,(t - nT_s)$$

$$X_p(w) = \frac{1}{2p}[X_a(w) * P(w)]$$

$$P(w) = \frac{2p}{T_s}\sum_{Ks-\infty}^{+\infty} d\,(w - kw_s)$$

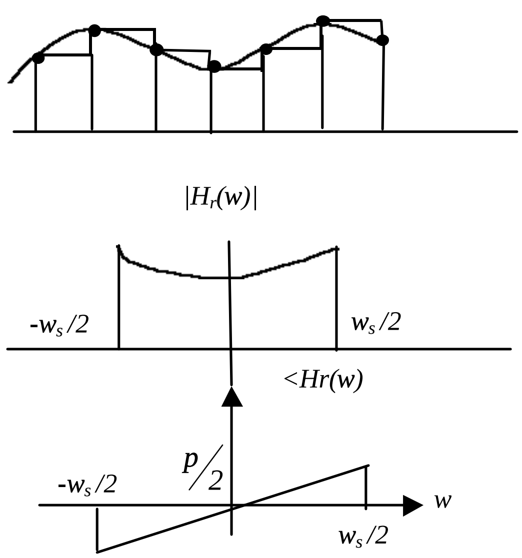
$$X_p(w) = \frac{1}{T_s}\sum_{-\infty}^{+\infty} X_a(w - kw_s)$$

Therefore, $X_p(w)$ is a periodic function of shifted the $X_a(w)$.



Obviously, if $w_s \geq 2w_M$ there is no aliasing and the signal can be reconstructed accurately.

In practice however, generating very narrow impulse is very difficult. Therefore, the practical way for sampling is zero-order hold. Such a system samples $x_a(t)$ at a given sampling instant and holds that value until the succeeding sampling instant. Reconstruction of $x_r(t)$ from the output of this system requires a cascade of low-pass filters or a non-constant gain of LPF.



It is as if $x_a(t) \rightarrow$