Lecture 6- Chapter 4

Frequency Analysis of Signals and Systems

Continuous Signals and Discrete-Time Signals

Periodic Aperiodic

Starting with periodic CT signals:

Recall that a linear combination of harmonically related complex exponentials of the form $x(t)=\sum_{k=-\infty}^{+\infty} C_k e^{j2\pi kF_0 t}$ is a periodic signal with fundamental period $T_p=\frac{1}{F_o}$. In order to find C_k, multiply both sides by $e^{-j2\pi\ell Fot}$ and integrate over one period:

$$\int_0^{T_P} x(t)e^{-j2\pi lF_0 t}dt=\int_0^{T_P} e^{-j2\pi lF_0 t}\sum_{k=-\infty}^{+\infty} C_k e^{j2\pi kF_0 t}dt=\sum_{k=-\infty}^{+\infty} C_k \underbrace{\int_0^{T_P} e^{-j2\pi(k-l)F_0 t}dt}_{\begin{cases}0 & k\neq l\\ T_P & k=l\end{cases}}$$

$$\Rightarrow \int_o^{Tp} x(t)e^{-j2\pi lFot}=C_\ell\cdot T_p$$

$$\therefore C_\ell=\frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi\ell Fot} \quad \text{Fourier Series}$$

An important issue is that whether $\sum_{k=0}^{+\infty} C_k e^{j2\pi kFot}$ representation is equal to *x(t)* for every moment of *t*. The Dirichlet conditions guarantee that this series is equal to *x(t)* except at the values of *t* for which *x(t)* is discontinuous. At those values of *t*, the series converges to the midpoint (average value) of the discontinuity.

Dirichlet conditions are:

$$\left.\begin{array}{l}\text{1) } x(t) \text{ has a finite number of discontinuity in any period.}\\ \text{2) } x(t) \text{ has a finite number of maxima and minima during each period}\\ \text{3) } x(t) \text{ is absolutely integrable in any period:}\end{array}\right\}\ \begin{array}{l}\text{sufficient but not}\\ \text{necessary}\end{array}$$

$\int_{Tp}|x(t)|<\infty$

A weaker condition is that signal's energy in one period should be finite: $\int_{Tp}|x(t)|^2 dt<\infty$

Power Density Spectrum of Periodic Signals

A periodic signal has infinite energy but finite average power.

Parseval's Theorem:

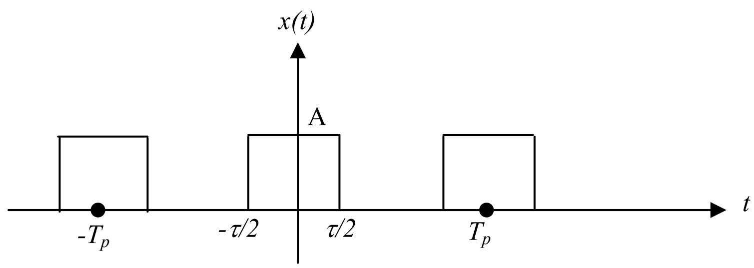
$$P_x = \frac{1}{T_p}\int_{Tp} |x(t)|^2 dt = \frac{1}{T_p}\int_{Tp} x(t)x^*(t)dt$$

$$= \frac{1}{T_p}\int_{Tp} x(t)\sum_{-\infty}^{+\infty} C_K^* e^{-j2\pi KFot}dt = \frac{1}{T_p}\sum_{-\infty}^{+\infty} C_K^* \underbrace{\int_{Tp} x(t)e^{-j2\pi KFot}dt}_{T_p C_K}$$

$$= \sum_{-\infty}^{+\infty} |C_K|^2$$

If *x(t)* is real then $C_k^* = C_{-k} \rightarrow |C_k|^2 = |C_k^*|^2 \Rightarrow PSD$ is an even function in frequency and the phase is an odd function.

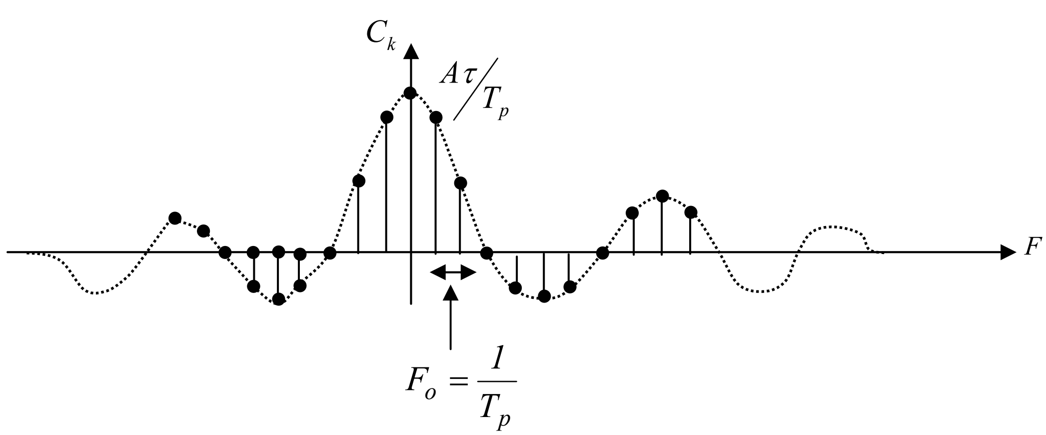
Example:



$$C_k = \frac{1}{T_p}\int_{-\tau/2}^{\tau/2} Ae^{-j2\pi kF_0 t}dt = \frac{A}{T_p}\left[\frac{e^{-j2\pi kFot}}{-j2\pi kF_0}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{\pi F_0 k T_p}\frac{e^{j\pi kF_0 t} - e^{-j\pi kF_0 \tau}}{2} = \frac{A\tau}{T_p}\frac{\sin(\pi kF_0\tau)}{\pi kF_0\tau}$$

$$= \frac{A\tau}{T_p}\sin c(\pi kF_0\tau)$$



Now if τ / T_p decreases $(T_p \to \infty)$, then $C_k \to 0$, which means the signal becomes aperiodic $\to$ average power becomes zero.

CT Aperiodic Signals

We can say $x(t) = \lim_{T_p \to \infty} x_p(t) \Rightarrow \sum \to \int$

$$x(t) = \int_{-\infty}^{+\infty} x(F) e^{j2\pi F t} dF = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\Omega) e^{j\Omega t} d\Omega \quad and$$

$$X(F) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi F t} dt = \int_{-\infty}^{+\infty} x(t) e^{-j\Omega t} dt$$

Aperiodic signals are energy signals.

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$
$$= \int_{-\infty}^{+\infty} |X(F)|^2 dF$$

Energy Density Spectrum: $S_{xx}(F) = |X(F)|^2$

A couple of points:

1) Remember that from only ESD or PSD we cannot reconstruct $x(t)$ because phase information is lost.
2) C_k for $x_p(t)$ is just samples of $X(F)$

$$C_k = \frac{1}{T_p} X(kF_o)$$

DT Frequency Analysis

First consider a periodic DT signal $x(n) = x(n + N)$

$$x(n) = \sum_{k=o}^{N-1} C_k e^{j2\pi \frac{k}{N} \cdot n}$$

Multiply both sides by $e^{-j2\pi \frac{\ell}{N} \cdot n}$ and sum over one period.

$$\sum_{n=o}^{N-1} x(n) e^{-j2\pi \frac{\ell}{N} n} = \sum_{n=o}^{N-1} \underbrace{\sum_{K=o}^{N-1} C_K e^{j2\pi \frac{(K-\ell)}{N} \cdot n}}$$

$$\sum_{n=o}^{N-1} a^n = \begin{cases} N, & \text{if } a = 1 \\ \dfrac{1-a^N}{1-a}, & \text{if } a \neq 1 \end{cases} \qquad = \begin{cases} N \; if \;\; K - \ell = o, \pm N, \pm 2N \\ o \qquad else \end{cases}$$

Interchange the sums

$$\Rightarrow \sum_{n=o}^{N-1} x(n) e^{-j2\pi \frac{\ell}{N} n} = N.C_\ell$$

$$C_k = \frac{1}{N} \sum_{n=o}^{N-1} x(n) e^{-j2\pi kn/N}, k = o, \ldots. N-1$$

$$x(n) = \sum_{K=o}^{N-1} C_k e^{j2\pi kn/N}$$

Power $\quad P_x = \sum_{k=o}^{N-1} |C_k|^2 = \frac{1}{N} \sum_{n=o}^{N-1} |x(n)|^2$

DTFS is periodic like Periodic DT

$C_{k+N} = C_{k.}$ Therefore, the spectrum of a periodic DT, *x(n),* is also periodic with period *N*.

$$C_{k+N} = \frac{1}{N} \sum_{n=o}^{N-1} x(n) e^{-j2\pi(k+N)n/N} = \frac{1}{N} \sum_{n=o}^{N-1} x(n) e^{-j2\pi kn/N} = C_k$$

Example

Find DTFS of the following signals:

$$x(n) = \underbrace{\cos \frac{2\pi}{3} n}_{\substack{x_1(n) \\ N_1 = 3}} + \underbrace{\sin \frac{2\pi}{5} n}_{\substack{x_2(n) \\ N_2 = 5}} \qquad N = 15 \text{ The smallest common denominator}$$

For this case, we can directly write it is a sum of exponentials.

$$x_1(n) = \frac{1}{2}\left[e^{j\frac{2\pi n}{3}} + e^{-j\frac{2\pi n}{3}} \right] = \frac{1}{2}\left[e^{j\frac{2\pi n}{3}} + \underbrace{e^{j\frac{(2-3)2\pi n}{3}}}_{e^{j\frac{2(2\pi)n}{3}}} \right]$$

$\rightarrow C_1 = \frac{1}{2}, \; C_2 = \frac{1}{2}$ for $x_1(n)$

$$x_2(n) = \sin\frac{2\pi}{5}n = \frac{1}{2j}\left[e^{j\frac{2\pi n}{5}} - e^{-j\frac{2\pi n}{5}}\right] = \frac{1}{2j}\left[e^{j\frac{2\pi n}{5}} - e^{-j\frac{(4-5)2\pi n}{5}}\right]$$

$\rightarrow C_1 = \frac{1}{2j}$, $C_4 = \frac{-1}{2j}$ for $x_2(n)$ and 0 else where.

C_k for $x(n)$ is like $C^{x_1}_{k\times 5} + C^{x_2}_{k\times 3}$

$$C_k = \begin{cases} \frac{1}{2j} & k = 3 \\ \frac{1}{2} & k = 5,10 \\ \frac{-1}{2j} & k = 12 \\ o & \text{else} \end{cases}$$

Fourier Transform for Aperiodic D.T. Signals

$$\begin{cases} X\left(e^{j\omega}\right) \equiv X(\omega) = \sum\limits_{n=-\infty}^{+\infty} x(n)e^{-j\omega n} \\ x(n) = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}\, d\omega \end{cases}$$

(recall that for C.T. signals it was over $\sum\limits_{-\infty}^{+\infty}$ and here is over 2π which means that $X(\omega)$ is periodic).

Two Basic Differences Between CTFT and DTFT:

1) $X(\omega) \equiv X(e^{j\omega})$ is periodic with period 2π

$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j(\omega+2k\pi)n} = \sum_{-\infty}^{+\infty} x(n)e^{-j\omega n} = X(\omega)$$

2) Since $X(\omega)$ is periodic, (in fact a it is a periodic C.T. signal), then it has a Fourier Series and in fact $x(n)$ are the coefficients of that Fourier Series.

Before visiting a famous example, let's review the concept of convergence.

If we have a limited observation, we will have the truncation effect, and the famous theory of the Gibbs. Let $X_N(\omega) = \sum\limits_{n=-N}^{N} x(n)e^{-j\omega n}$ if $\underbrace{\lim |X_N(\omega) - X(\omega)|}_{N\rightarrow\infty} \rightarrow 0$, then $X_N(\omega)$ converges uniformly to *X(ω)* as $N \to \infty$. This convergence is guaranteed if *x(n)* is absolutely summable (3rd Dirichlet condition).

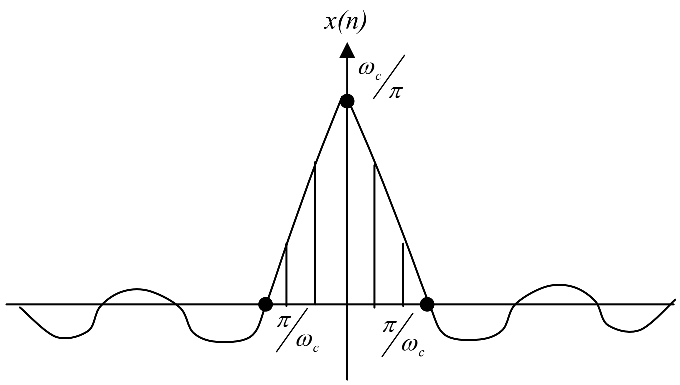
$\sum_{-\infty}^{+\infty}|x(n)| < \infty$ this implies $|X(\omega)| = \left|\sum_{-\infty}^{+\infty} x(n)e^{-j\omega n}\right| < \sum_{-\infty}^{+\infty}|x(n)| < \infty$, which means *X(ω)* exists and is somehow band limited and hence, the uniform convergence.

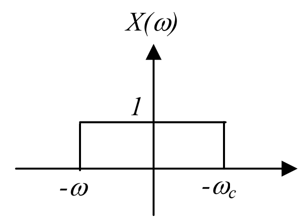
However, this is a sufficient condition. If *x(n)* is not absolutely summable but square summable (finite energy) then *X(ω)* can exist.

If $E_x = \sum_{-\infty}^{+\infty}|x(n)|^2 < \infty$, there is not a uniform convergence but there is a mean-square convergence.

$$\lim_{N\to\infty}\int_{-\infty}^{+\infty}|X(\omega) - X_N(\omega)|^2\, d\omega = 0 = \lim_{N\to\infty} E(error) \to 0$$

Meaning the energy of error goes to zero but not necessarily the error itself.





The example of this particular case is the sinc function.

$$x(n) = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

This is not absolutely summable. Hence, the $X_N(\omega) = \sum_{-N}^{N} x(n)e^{-j\omega n}$ doesn't converge to *X(ω)* uniformly for all *ω*. However, *x(n)* has a finite energy $E_x = \frac{\omega_c}{\pi}$. So $X_N(\omega)$ converges to *X(ω)* in mean square sense.

$$X_N(\omega) = \sum_{-N}^{N} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

Matlab definition: $\operatorname{sin}c(x) = \dfrac{\sin \pi x}{\pi x}$

Section 4.2.12

There are two time-domain characteristics that determine the type of signal spectrum and they are: Periodicity and Continuity

Signals

- Continuous
  - Periodic: Aperiodic & Discrete Spectrum
  - Aperiodic: Aperiodic & Continuous Spectrum
- Discrete-Time
  - Periodic: Periodic & Discrete Spectrum
  - Aperiodic: Periodic & Continuous Spectrum

** Periodicity with period α in one domain automatically implies discretion with spacing $1/\alpha$ in the other domain.

__**Properties of the Fourier Transform for Discrete-Time Signals**__

1. Real signals – if x(n) is real, the $X^*(\omega) = X(-\omega)$

   Spectrum magnitude: $|X(\omega)| = |X(-\omega)|$ $\rightarrow$even function

   Spectrum phase $\measuredangle X(-\omega) = -\measuredangle X(\omega) \rightarrow$ odd function.
2. Real and even $x(n)$ $\rightarrow$ Real and Even $X(\omega)$
3. Real and odd $x(n)$ $\rightarrow$ Imaginary and odd $X(\omega)$
4. Imaginary and odd $x(n)$ $\rightarrow$ Real and odd $X(\omega)$
5. Imaginary and even $x(n)$ $\rightarrow$ Imaginary and even $X(\omega)$
6. Linearity $a_1x_1(n) + bx_2(n) \overset{F}{\Longleftrightarrow} aX_1(\omega) + bX_2(\omega)$
7. Time-Shifting $x(n) \overset{F}{\Longleftrightarrow} X(\omega)$

   $x(n-k) \overset{F}{\Longleftrightarrow} e^{-j\omega k} X(\omega)$

8. Time-Reversal $x(n) \overset{F}{\longleftrightarrow} X(\omega)$

$x(-n) \overset{F}{\longleftrightarrow} X(-\omega)$ Therefore, FT of an even function is an even function too.

9. Convolution: $x_1(n) * x_2(n) \overset{F}{\longleftrightarrow} X_1(\omega).X_2(\omega)$

Proof:

$$x(n) = \sum_{k=-\infty}^{+\infty} x_1(k)x_2(n-k)$$

$$X(\omega) = \sum_{n=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} x_1(k)x_2(n-k)\underbrace{e^{-j\omega n}}_{e^{-j\omega(n-k)}.e^{-j\omega k}}$$

$$= \sum_{k=-\infty}^{+\infty} x_1(k)e^{-j\omega k}\sum_{n=-\infty}^{+\infty} x_2(n-k)e^{-j\omega(n-k)}$$

$$= X_1(\omega).X_2(\omega)$$

10. Correlation Theorem: $r_{x_1x_2}(n) \overset{F}{\longleftrightarrow} X_1(\omega)X_2(-\omega)$

Proof:

$$r_{x_1x_2}(n) = \sum_{k=-\infty}^{+\infty} x_1(k)x_2(k-n)$$

Cross-Energy Density Spectrum

FT

$$S_{x_1x_2}(\omega) = \sum_{n=-\infty}^{+\infty} r_{x_1x_2}(n)e^{-j\omega n} = \sum_{n=-\infty}^{+\infty}\sum_{K=-\infty}^{+\infty} x_1(K)x_2(K-n)\underbrace{e^{-j\omega n}}_{e^{-j\omega(n-K)}\cdot e^{-j\omega K}}$$

$$RHS = \underbrace{\sum_{K=-\infty}^{+\infty} x_1(K)e^{-j\omega K}}_{x_1(\omega)}\ \underbrace{\sum_{n=-\infty}^{+\infty} x_2[-(n-K)]e^{-j\omega(n-K)}}_{x_2(-\omega)}$$

Now if $x(n)$ is real, then $X^*(\omega) = X(-\omega)$

$$r_{xx}(\ell) \leftrightarrow S_{xx}(\omega) = x(\omega)\underbrace{x(-\omega)}_{x^*(\omega)} = |x(\omega)|^2$$

Energy Spectral Density

11. Frequency Shifting

$$e^{j\omega on} \overset{F}{\longleftrightarrow} X(\omega - \omega_o)$$

12 Modulation Theorem

$$x(n)\cos(\omega_0 n) \leftrightarrow \frac{1}{2}\left[X(\omega+\omega_o)+X(\omega-\omega_o)\right]$$

$$\cos(\omega_0 n) = \frac{1}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right]$$

13. Parseval Theorem

$$\sum_{n=-\infty}^{+\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)\,d\omega$$

Proof:

$$RHS = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty} x_1(n)e^{-j\omega n}\right]X_2^*(\omega)\,d\omega$$

$$= \sum_{n=-\infty}^{\infty} x_1(n)\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi} X_2^*(\omega)e^{-j\omega n}d\omega}_{x_2^*(n)} = \sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n)$$

Special case: $x_2(n) = x_1(n) \rightarrow \Sigma|x(n)|^2 = \frac{1}{2\pi}\int_{2\pi}|x(\omega)|^2 d\omega$

$$E_x = r_{xx}(0) = \sum_{n=-\infty}^{+\infty}|x(n)|^2 = \frac{1}{2\pi}\int_{2\pi}\underbrace{|x(\omega)|^2 d\omega}_{S_{xx}(\omega)d\omega}$$

14. Windowing

$$x_1(n)\cdot x_2(n) \quad \overset{F}{\Leftrightarrow} \quad \frac{1}{2\pi}\left[X_1(\omega) * X_2(\omega)\right]$$

$$x(\omega) = \sum_{-\infty}^{+\infty} x(n)e^{-j\omega n} = \sum_{-\infty}^{+\infty} x_1(n)x_2(n)e^{-j\omega n}$$

$$RHS = \sum_{-\infty}^{+\infty}\left[\frac{1}{2\pi}\int_{2\pi} X_1(\lambda)e^{j\lambda n}d\lambda\right]x_2(n)e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{2\pi} X_1(\lambda)d\lambda \sum_{n=-\infty}^{+\infty} x_2(n)e^{j\lambda n}\cdot e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{2\pi} X_1(\lambda)X_2(\omega-\lambda)d\lambda \equiv \frac{1}{2\pi}\left[X_1(\omega) * X_2(\omega)\right]$$

15. Differentiation in Frequency Domain

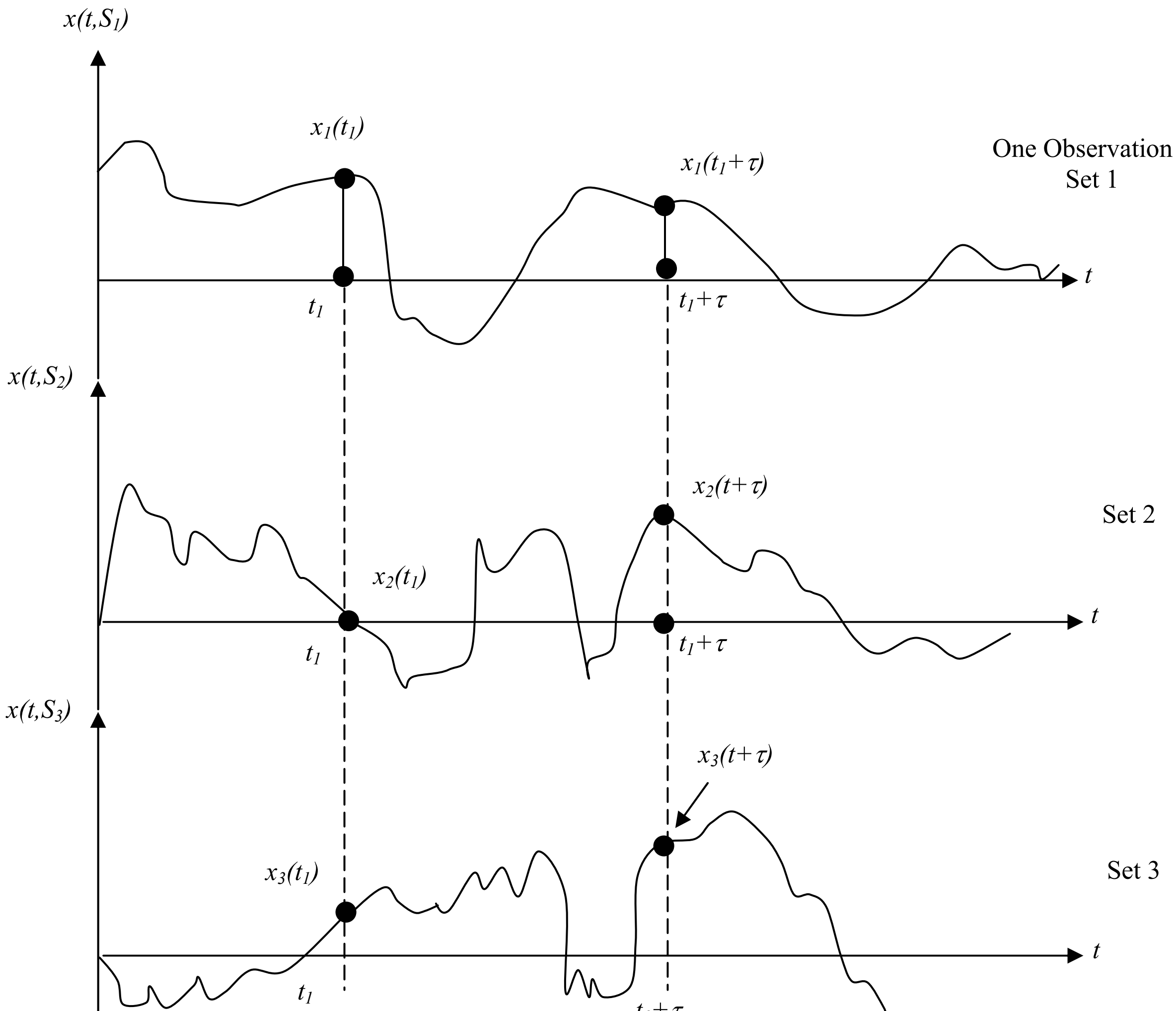
$$n(x)n \quad \overset{F}{\Leftrightarrow} \quad j\frac{dx(\omega)}{d\omega}$$

Skipping to Section 4.4.8

Correlation Function and Power spectra for Random Input Signals

When the input signal is random, then we have to consider statistical moments of input and output. So here is a bit of introduction about "Stationary Random Process". Starting with the Definition of Stationary Signals:

If *X(t)* is a random process with a point Probability Density Function (PDF), $P(x) = P(x_{t1}, x_{t_2}, x_{t3}, ..., x_t n)$ for *n* random variables. $X(t_i) \equiv x(t_i), i_{I1,2,...n}$



If the joint probability of $P(x_1, x_{,1}....., x_n)^{t_1} = P(x_1,....x_n)^{t_1+\tau}$ for all t_1 and τ then the random process *X(t)* is stationary in strict sense. In other words, statistical properties of a stationary random process is time-invariant, meaning that its mean and variance and other moments are time invariant.

Statistical (ensemble) Average

$$E\left(x_{t_i}\right)=\int_{-\infty}^{+\infty} x_{t_i} P\left(x_{t_i}\right) dx_{t_i}$$

If we don't have $P(x_{ti})$ but have many observations, then $E\left(x^{t_1}\right)=\frac{1}{N}\left(x_1^{t_1}+x_2^{t_1}+\ldots+x_N^{t_1}\right)$, which of a stationary process it is equal to $E(x^{ti})$ for any t_i.

Also autocorrelation function:

$$\begin{aligned}\gamma_{xx}\left(x^{t1},x^{t2}\right) &= \int\int_{-\infty}^{+\infty} x^{t1}x^{t2}P\left(x^{t1},x^{t2}\right)dx^{t1}dx^{t2} \\ &= E\left[x^{t1}\cdot x^{t2}\right]\end{aligned}$$

If this γ_{xx} depends only on the time difference $t_1 - t_2 = \tau$, then $\gamma_{xx}(\tau)=E\left[x_{t1},x_{t1+\tau}\right]$, which is the case for stationary process. If a process has two features:

1) Its γ_{xx} depends only on time difference, τ, and
2) $E(x^{t1}) = E(x^{t2}) = E(x^{ti})$

then the process is said to be *stationary in "wide sense"*.

Now if all statistical averages can be obtained by one single realization (or one sample set), then the process is also *"Ergodic"* meaning *Ensemble average* $\equiv$ *time average.*

$$\mu_x = E\left(x_n\right) \text{ and } \hat{\mu}_2 = \frac{1}{N}\sum_{n=o}^{N-1} x(n)$$

$\hat{\mu}_x$ is an estimate of μ_x. It will be said it is an unbiased estimate if $E\left(\hat{\mu}_x\right)=\mu_x$. Also, it is a good estimator if

$$Var\left(\hat{\mu}_x\right)=E\left(\left|\hat{\mu}_x\right|\right)^2-\left|\mu_x\right|^2 \rightarrow 0 \text{ as } N\rightarrow\infty$$

Therefore, time average $\rightarrow$ ensemble average.

Autocorrelation: $\gamma_{xx}(m)=\frac{1}{N}\sum_{n=o}^{N-1} x^*(n)x(n+m)$ $\quad E\left[\gamma_{xx}(m)\right]=r_{xx}(m)$ the true autocorrelation.

Now back to systems:

$$x(n)\rightarrow \boxed{h(n)} \rightarrow y(n)$$

$$\mu_y = E\{y(n)\} = E\left\{\sum_{n=-\infty}^{+\infty} h(k)x(n-k)\right\}$$

$$= \sum_{-\infty}^{+\infty} h(k)E\{x(n-k)\} = \mu_x \sum_{-\infty}^{+\infty} h(k) = \mu_x H(0)$$

The autocorrelation sequence:

$$\gamma_{yy}(m) = E\{y^*(n)y(n+m)\} = E\left\{\sum_{k=-\infty}^{+\infty} h(k)x^*(n-k)\cdot \sum_{\ell=-\infty}^{+\infty} h(\ell)x(n+m-\ell)\right\}$$

$$= \sum_{k}^{\infty}\sum_{\ell}^{\infty} h(k)h(\ell)E\{x^*(n-k).x(n+m-\ell)\}$$

$$= \sum_{k}^{\infty}\sum_{\ell}^{\infty} h(k)h(\ell)\gamma_{xx}(m-\ell+k)$$

Special Form: when *x(n)* is a white noise, then $\gamma_{xx}(m) = \sigma_x^2 \delta(m)$ and $\sigma_x^2 = \gamma_{xx}(0)$. Then

$$\gamma_{yy}(m) = \sigma_x^2 \gamma_{hh}(m)$$

$$\gamma_{yy}(0) = \sigma_x^2 \gamma_{hh}(0) = \sigma_x^2 \frac{1}{2\pi}\int_{2\pi} |H(\omega)|^2 d\omega$$

by getting the Fourier transform in general:

$$\Gamma_{yy}(\omega) = \sum_{m=-\infty}^{\infty} \gamma_{yy}(m)e^{-j\omega m}$$

$$= \sum_{m=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} h(k)h(l)\gamma_{xx}(m-l+k)\right]e^{-j\omega m}$$

let $u = m-l+k \Rightarrow e^{-j\omega m} = e^{-j\omega u}.e^{-j\omega l}.e^{j\omega k}$

$$\Gamma_{yy}(\omega) = \sum_{k} h(k)e^{j\omega k}\sum_{l} h(l)e^{-j\omega l}\sum_{u}\gamma_{xx}(u)e^{-j\omega u}$$

$$= H(-\omega).H(\omega).\Gamma_{xx}(\omega)$$

If the signal is real $= \Gamma_{xx}(\omega)|H(\omega)|^2$