Lecture 4

Properties of ROC

1) ROC is a ring $o \le r_1 < |z| < r_2 \le \infty$
2) The Fourier Transform of *x(n)* converges if and only if ROC of *X(z)* includes the unit circle.

   (Remember, $z = re^{j\omega}$ and if $|z| = 1$ then $r = 1$ and then $X(z) = X(e^{j\omega}) = \sum_{-\infty}^{+\infty} x(n) e^{-j\omega n} =$

   Fourier Transform of *x(n)*. So, if *X(z)* convergence region includes the unit circle, then $X(e^{j\omega}) = X(\omega)$ exists.)
3) ROC cannot contain any poles.
4) If *x(n)* is finite, then ROC is the entire plane except $z = o/\infty$
5) If *x(n)* is right-sided (i.e., $x(n) = o$ *for* $n < N_1, < \infty$) ROC is the exterior of the largest pole.
6) If *x(n)* is the left-sided (i.e. $x(n) = $ o for $n > N_2 > -\infty$) then ROC is the innermost ring of the smallest pole.
7) If *x(n)* is two-sided, ROC consists of a ring in *z* plane, bounded on the interior and exterior by a pole and not containing any pole.
8) ROC must be a connected region.

Properties of the Z-Transform

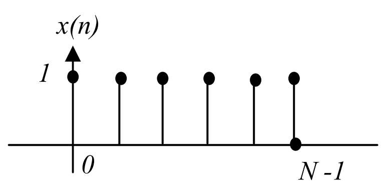
Time Shifting: $x(n-k) \xleftrightarrow{Z} z^{-k} X(z)$

Linearity: $ax_1(n) + bx_2(n) \xleftrightarrow{Z} aX_1(z) + bX_2(z)$

But we cannot say ROC = $ROC_1 + ROC_2$

Example 1:

$$x(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & else \end{cases}$$



Direct Method:

$$X(z) = \sum_{0}^{N-1} x(n) z^{-n}$$

$$= \sum_{0}^{N-1} z^{-n} = \begin{cases} N & if \ z = 1 \\ \dfrac{1 - z^{-N}}{1 - z^{-1}} & if \ z \ne 1 \end{cases}$$

The function $\frac{1-z^{-n}}{1-z^{-1}} = \frac{z^N - 1}{z^{N-1}(z-1)}$ has two poles at *0* and *1,* but $z = 1$ is not a pole for *X(z)* because it is defined to be *1* at $z = 1$. Therefore, ROC ≡ the entire plane except $z = 0$.

Now using *Z*-Transform properties:

$$x(n) = u(n) - u(n - N)$$

$$X(z) = \left(1 - z^{-N}\right)U(z) = \left(1 - z^{-N}\right)\frac{1}{1 - z^{-1}}$$

and ROC of this one is $|z| > 1$ while that is different from ROC found earlier.

So, if the linear combination of several signals has finite duration, the ROC of its z-transform is exclusively dictated by the finite duration of this signal and not by the ROC of the individual transforms.

<u>Scaling</u> $a^n x(n) \leftrightarrow X\left(\frac{z}{a}\right)$ ROC: $|a|\, r_1 < |z| < |a|\, r_2$

<u>Time-Reversal</u> $x(-n) \leftrightarrow X\left(z^{-1}\right)$ ROC: $\frac{1}{r_2} < |z| < \frac{1}{r_1}$

<u>Differentiation</u> $nx(n) \leftrightarrow -z\frac{dX(z)}{dz}$ same ROC

Example 2:

Determine *x(n)* if $X(z) = \log\left(1+az^{-1}\right)$ and $|z| > |\alpha|$.

$$\frac{dX(z)}{dz} = \frac{-az^{-2}}{1+az^{-1}} \Rightarrow -z\frac{dX(z)}{dz} = \frac{az^{-1}}{1+az^{-1}} = az^{-1}\left[\frac{1}{1-\left(-az^{-1}\right)}\right]$$

$$\because (-a)^n u(n) \xrightarrow{Z} \left[\frac{1}{1-\left(-az^{-1}\right)}\right] \Rightarrow InvZ\left\{\frac{az^{-1}}{1-\left(-az^{-1}\right)}\right\} = a(-a)^{n-1}u(n-1) \equiv nx(n)$$

$$\Rightarrow x(n) = (-1)^{n-1}\frac{a^n}{n}u(n-1)$$

<u>Convolution</u>

$$x_1(n) * x_2(n) \xleftrightarrow{Z} X_1(z)\cdot X_2(z)$$

ROC is at least the intersection of that for $X_1(z)$ and $X_2(z)$.

<u>Correlation</u>

$r_{x1x2}(\ell) \xleftrightarrow{Z} X_1(z)\cdot X_2\left(z^{-1}\right)$ remember that $r_{x1x2}(\ell) = x_1(\ell) * x_2(-\ell)$

$x^*(n) \leftrightarrow x^*(z^*)$

Time Multiplication

$$x_1(n)\cdot x_2(n) \overset{Z}{\longleftrightarrow} ?$$

$$X(z)=\sum_{n=-\infty}^{+\infty} x_1(n)x_2(n)z^{-n} = \sum_{n=-\infty}^{+\infty}\left[\frac{1}{2\pi j}\oint X_1(v)v^{n-1}dv\right]x_2(n)z^{-n}$$

$$=\frac{1}{2\pi j}\oint_c X_1(v)\left[\sum_{n=-\infty}^{+\infty} x_2(n)v^{+n}z^{-n}\right]v^{-1}dv$$

$$=\frac{1}{2\pi j}\oint X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$$

Parseval's Theorem

$$\sum x_1(n)x_2^*(n)=\frac{1}{2\pi j}\oint X_1(v)X_2^*\left(\frac{1}{v*}\right)v^{-1}dv$$

It is like evaluating $Z\{x_1(n)x_2^*(n)\}$ at $z = 1$ circle.

Initial Value Theorem

If $x(n)$ is causal, $x(0)=\lim_{z\to\infty} X(z)$

Proof: $X(z)=\sum_{o}^{\infty} x(n)z^{-n} = x(o)+x(1)z^{-1}+x(2)z^{-2}+\ldots$

if $z\to\infty$ $z^{-n}\to o$ therefore, $x(0)=\lim_{z\to\infty} X(z)$.

Example:

Using Z-transform properties, find $X(z)$ of the following signal.

$$x(n)=(n-2)0.5^{n-2}\cos\left[\frac{\pi}{3}(n-2)\right]u(n-2)$$

$$X(z)=z^{-2}Z\left\{\left[n(0.5)^n\cos\frac{\pi n}{3}\right]u(n)\right\}$$

$$=z^{-2}\left[-z\frac{d}{dz}Z\left\{\left[(0.5)^n\cos\frac{\pi n}{3}\right]u(n)\right\}\right]$$

$$Z\left\{(0.5)^n\cos\frac{\pi n}{3}u(n)\right\}=\frac{1-\left(0.5\cos\frac{\pi}{3}\right)z^{-1}}{1-2\left(0.5\cos\frac{\pi}{3}\right)z^{-1}+0.25z^{-2}}$$ From Table on page 174

$$=\frac{1-0.25z^{-1}}{1-0.25z^{-1}+0.25z^{-2}};\ |z|>0.5$$

$$X(z) = -z^{-1}\frac{d}{dz}\left\{\frac{1-0.5z^{-1}}{1-0.5z^{-1}+0.25z^{-2}}\right\}$$

$$X(z) = \frac{0.25z^{-3}-0.5z^{-4}+0.0625z^{-5}}{1-z^{-1}+0.75z^{-2}-0.25z^{-3}+0.0625z^{-4}}; \qquad |z| > 0.5$$

Now, lets use MATLAB to see if we've computed correctly.

```
b = [0, 0, 0, 0.25, - 0.5, 0.0625];
a = [1, -1, 0.75, -0.25, 0.1625];
n = 0:  20                                    % checking the fist 21 samples of x(n)
delta = [n =0];                               % creating δ(n)
x = filter (b, a, delta),
plot (n, x), hold
x = [zeros (1, 2)  n.* (0.5.^n) * cos(pi * n/3)];          % creating the original signal
n1 = 0:22;
plot (n1, x, 'r')
```

Rational *Z*-Transforms

$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$ if a_o and $b_o \neq 0$, then we can rewrite it as: $X(z) = \frac{b_0}{a_0} z^{-M+N} \frac{\prod_{k=1}^{M}(z-z_k)}{\prod_{k=1}^{N}(z-p_k)}$

It has M finite zeros at $z_1, z_2, \ldots, z_M$ and N finite poles at $p_1, p_2, \ldots, p_N$ as well as $N - M$ zeros or $M - N$ poles at origin and a possible zero/pole at ∞. Depending on the location of the poles, the signal has different behaviors. Read Section 3.3.2.

The System Function of a LTI System

$Y(z) = H(z) \cdot X(z)$ $H(z)$ is called the system-function. A system in general can be presented by a difference equation:

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$
$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{N} b_k z^{-k} X(z)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=o}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$, where $a_0 = 1$

Special Cases:

If $a_k = 0$ for $1 \leq k \leq N$, then $H(z)=\sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M}\sum_{k=0}^{M} b_k z^{M-k}$, which is an all-zeros system.

The system has M trivial poles at the origin. Such a system has a finite duration impulse response and therefore is called FIR system.

On the other hand, if $b_k = 0$ for $1 \leq k \leq M$ then $H(z)=\frac{b_0}{1+\sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 z^N}{\sum_{K=0}^{N} a_k z^{N-k}}, a_0 = 1$.

This system is an all-pole system (has N trivial zeros at origin) and therefore, has an infinite duration impulse response and thus is called IIR system. A pole-zero system is still IIR because of the poles.

The Inverse of Z Transform

$$X(z)=\sum_{k=-\infty}^{+\infty} x(k) z^{-k}$$

By multiplying both sides of the above formula by z^{n-1} and integrating both sides over a closed contour within ROC of $X(z)$, which encloses the origin, we have:

$$\oint_c X(z) z^{n-1} dz = \oint_c \sum_{k=-\infty}^{+\infty} x(n) z^{n-1-k} dz$$

Since the series converges on this contour, we can interchange Σ and $\oint$. Then

$$\oint_c X(z) z^{n-1} dz = \sum_{k=-\infty}^{\infty} x(k) \underbrace{\oint_c z^{n-k-1} dz}_{=\begin{cases}2\pi j & n=k\\ 0 & n\neq k\end{cases}}$$ Cauchy Integral Theorem

$$\boxed{x(n)=\frac{1}{2\pi j}\oint_c x(z) z^{n-1} dz}$$

One of Caushy Theorems states $\oint_c (z-z_o)^n dz = \begin{cases} o & n \neq -1 \\ 2\pi j & n=-1 \end{cases}$

Let $z_o = o$. Then, $f(z) = z^n$. If n is positive the antiderivitive $\frac{z^{n+1}}{n+1}$ is analytic every where and therefore, its contour integral is zero. But only for $f(z) = z^{-1}$ it doesn't have an antiderivitive even in a punctured plane. For $n \le -2$, it is analytic in a punctured plane with origin deleted.

Remember that if f is analytic in a simply connected domain, D, and Γ is any loop (close contour) in D, then $\int_{\Gamma} f(z)dz = 0$ because in a simply connected domain any loop can be shrunk to a point. Therefore, the integral of a continuous function over a shrinking loop converges to zero.