Lecture 2 (Chapter 2) Discrete-Time Signal and Systems

<u>Classification of Signals</u>

1. Finite duration $x(n) = 0 \quad n > N$

   Infinite duration

2. Right-Left sided

   $x(n) = 0 \qquad n < N_1$ right-sided

   $x(n) = 0 \qquad n > N_2$ left-sided

Some Elementary Discrete-Time Signals

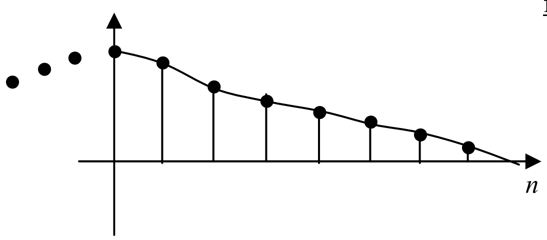
- unit sample sequence $\sigma(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$
- unit step sequence $u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$
- unit ramp $u_r(n) = \begin{cases} n & n \geq 0 \\ 0 & n < 0 \end{cases}$
- exponential $x(n) = a^n$ for all n

If a is complex, then $a = re^{j\theta} \rightarrow x(n) = r^n e^{j\theta n}$

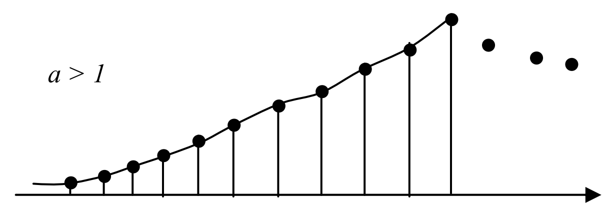
$$= r^n(\cos\theta n + j\sin\theta n)$$

● < a< 1

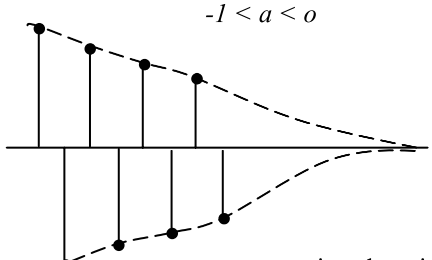
<u>for real *a*</u>



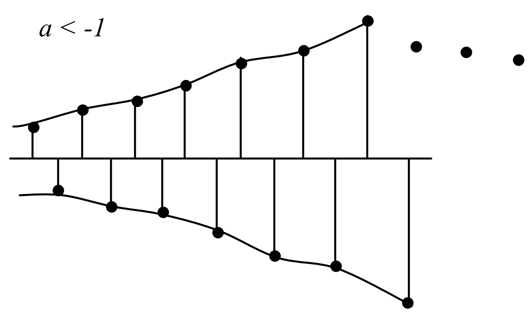
a > 1



-1 < a < o



a < -1

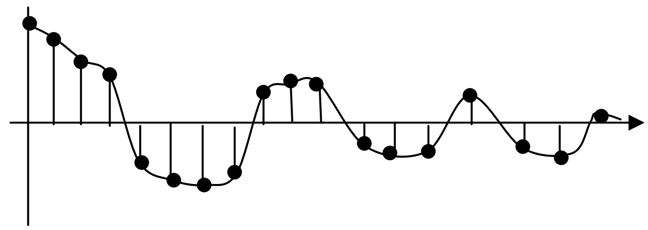


r is a damping factor

For complex a, $x(n) = x_R(n) + jx_I(n)$, where $x_R(n) = r^n \cos(\theta n)$

if r < 1 then

$$\begin{cases} |x(n)| = r^n \\ \angle x(n) = \theta n \end{cases}$$



Energy and Power Signals

Energy is defined as $E = \sum_{-\infty}^{+\infty} |x(n)|^2$ if E is finite, i.e., $o < E < \infty$, then *x(n)* is called Energy Signal. However, many signals that have an infinite energy, have a finite average power. Average power is defined as

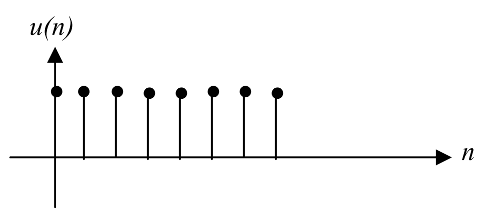
$$P_{ave} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{-N}^{N} |x(n)|^2 .$$

If we define the signal energy of *x(n)* over the interval *(-N, N)* as

$$E_N = \sum_{-N}^{N} |x(n)|^2 \quad \text{then} \quad E = \lim_{N \to \infty} E_n$$

and therefore, $P_{ave} = \lim_{N \to \infty} \frac{1}{2N+1} E_N$ clearly if E is finite, then $P_{ave} = 0$.

<u>Example</u> – Unit Step Sequence



Obviously, it is not an energy signal but it is a power signal.

$$p = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} \left|x(n)\right|^2 = \lim \frac{1}{2N+1} \sum_{n=0}^{N} (1)^2$$

$$= \lim_{N\to\infty} \frac{N+1}{2N+1} = \frac{1}{2} \Rightarrow \quad \text{it is a power signal!}$$

<u>Periodic and Aperiodic Signals</u>

$$x(n+N) = x(n) \Rightarrow \text{signal is periodic}$$

Energy of periodic signals is infinite but it might be finite over a period. On the other hand, the average power at the periodic signal is finite and is equal to the P_{ave} over a single period. Hence, periodic signals are power signals.

<u>Symmetric (even) and odd Signals</u>

$$x(n) = x(-n) \text{ even}$$

$$x(n) = -x(-n) \text{ odd}$$

Any signal can be written as: $x(n) = x_e(n) + x_o(n)$

Where $\begin{cases} x_e(n) = \frac{1}{2}\left[x(n) + x(-n)\right] \\ x_o(n) = \frac{1}{2}\left[x(n) - x(-n)\right] \end{cases}$

Read Section 2.1.8

Classification of Discrete-Time Systems

<u>Static versus Dynamic Systems</u>

Static Systems $\equiv$ memory less $\equiv$ the output doesn't depend on past or future values of the input.

Dynamic Systems $\equiv$ having either infinite or finite memory.

Example: $y(n) = 2x(n) + x(n)^2$ Static

$$y(n) = \sum_{k=0}^{N} x(n-k) \text{ Dynamic-finite}$$

$$y(n) = \sum_{k=0}^{\infty} x(n-k) \text{ Dynamic-infinite}$$

Time invariant versus Time-Invariant Systems

A relaxed system Γ is time-invariant if

$$x(n) \rightarrow y(n)$$
$$\forall K, x(n-k) \rightarrow y(n-k)$$

Example: 1) $y(n) = x(n) - x(n-1)$

$y(n-k) = x(n-k) - x(n-k-1)$ time invariant

2) $y(n) = nx(n)$

$y(n-k) = (n-k)\,x(n-k) = nx(n-k) - kx(n-k)$

but $x(n-k) \rightarrow nx(n-k) \neq y(n-k)$

$\Rightarrow$ time variant

Causality

A system is causal if the output at any time depends only on present and past values of the inputs and not on future values of the input. $y(n) = x(-n)$ is non-casual because $y(-1) = x(1)$!

Stable versus Unstable Systems

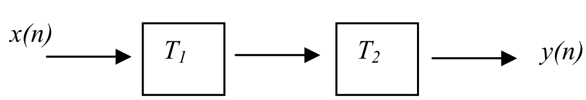
A system is *Stable* if any bounded input produces bounded output (BIBO).
Otherwise, it is *unstable*.

Linearity

A system is linear if

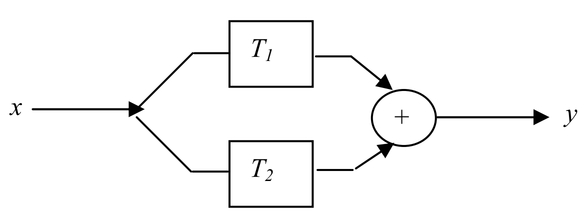
$$T(a_1x_1(n) + a_2x_2(n)) = a_1T[x_1(n)] + a_2T[x_2(n)]$$

two systems can be connected to each other in two ways:



$T = T_1 \bullet T_2 = T_2 \bullet T_1$

= for LTI systems only



$T = T_1 + T_2$

LTI Systems

LTI systems are the most important class of systems because the behavior of the system is known by knowing its response to unit sample input (*x(n)*).

$$y(n) = F\{x(n)\} = F\left[\sum_{K=-\infty}^{+\infty} x(K)\sigma(n-K)\right]$$

Then
$$= \sum_{K=-\infty}^{+\infty} x(K) F \underbrace{[\sigma(n-K)]}_{h(n-K)} = \sum_{K=-\infty}^{+\infty} x(K) h(n-K)$$

$$\Rightarrow y(n) = x(n) * h(n) = h(n) * x(n)$$
$$= \sum_{K} x(K) h(n-K) = \sum_{K} h(K) x(n-K)$$

Causality of LTI System

$$y(n) = \sum_{-\infty}^{\infty} h(k)x(n-k) = \sum_{-\infty}^{-1} h(k)x(n-k) + \sum_{0}^{\infty} h(k)x(n-k)$$
$$= \underbrace{[... + h(-2)x(n+2) + h(-1)x(n+1)]}_{} + [h(0)x(n) + h(1)x(n-1) + ...]$$

this depends on future values of *x(n)*

Hence, for a system to be casual, its *h(n)* must be zero for *n* < *0*.

Stability of LTI Systems

The BIBO system means:

$$|x(n)| < M_x < \infty \rightarrow |y(n)| M_y < \infty$$
$$y(n) = \sum_{k=-\infty}^{+\infty} h(k) x(n-k)$$
$$|y(n)| = \left| \sum_{k=-\infty}^{+\infty} h(k) x(n-k) \right| \le \sum_{k=-\infty}^{+\infty} |h(k)| |x(n-k)| \le \sum_{k} |h(k)| M_x$$

Therefore if $\sum_{k} |h(k)| < \infty$ meaning *h(k)* is absolutely summable, then $|y(n)| < \infty \Rightarrow$ the system is stable. This condition is not only sufficient for stability of the system, but it is also necessary. Indeed we should show that if $\sum_{k} |h(k)| = \infty$, then a bounded input can produce an unbounded output. Let

$$x(n)=\begin{cases}\dfrac{h^*(-n)}{\left|h^*(-n)\right|} & h(n)\neq 0\\ 0 & h(n)=0\end{cases}$$ then $y(n)=\sum_k h(k)x(n-k)$ and

$$y(o)=\sum_{-\infty}^{+\infty}h(k)x(-k)=\sum_k\frac{\left|h(k)\right|^2}{h(k)}=\sum\left|h(k)\right|=\infty\ \to\ y(0)=\infty\equiv \text{unbounded.}$$

The condition $\sum_k\left|h(k)\right|<\infty$ also implies that *h(n)* goes to zero as *n* approaches $\infty\Rightarrow$ the output of the system goes to zero as $n\to\infty$ if the input is set to zero beyond $n > n_o$. In other words, a finite excitation to the system creates an output that eventually dies out (transient response).

Let's prove it. Suppose $\left|x(n)\right|<M_x$ and also $x(n) = 0$ for $n > n_o$. Then at any $n = n_o + N$, the system output is:

$$y(n_o+N)=\underbrace{\sum_{k=-\infty}^{N-1}h(k)x(n_o+N-k)}_{=0}+\sum_{k=N}^{\infty}h(k)x(n_o+N-k)$$

$$\because x(n)=0 \text{ for } n\geq n_o$$

Hence, $\left|y(n_o+N)\right|=\left|\sum_{k=N}^{\infty}h(k)x(n_o+N-k)\right|\leq\sum_{K=N}^{\infty}\left|h(k)\right|\left|xk\right|\leq M_x\sum_{k=N}^{\infty}\left|h(k)\right|$

Now as $N\to\infty\ \ell im_{N\to\infty}\sum_{n=N}^{\infty}\left|h(k)\right|=0\ \ \Rightarrow \ell im_{N\to\infty}y(n_o+N)=0$ Therefore, the transient response goes to zero and hence the system is stable.

Example: $h(n) = a^n u(n)$ determine the range that *h(n)* is stable.

1) System is causal for $h(n) = 0$ for $n < 0$

2) $\sum_{k=-\infty}^{\infty}\left|h(k)\right|=\sum_{k=0}^{\infty}\left|a^k\right|=1+\left|a\right|+\left|a^2\right|+\ldots$

Now if $|a| < 1$ this converges to $\dfrac{1}{1-\left|a\right|}$ and the system would be stable but if $|a| > 1$ the system is unstable.

If $h(n) = 0$ for $n \geq M$ and $n < 0$, then the system is called FIR (Finite duration Impulse Response); otherwise the system if called IIR.

FIR system has a finite memory and it only looks at the input within a window.

<u>Analyzing the FIR and IIR System</u>

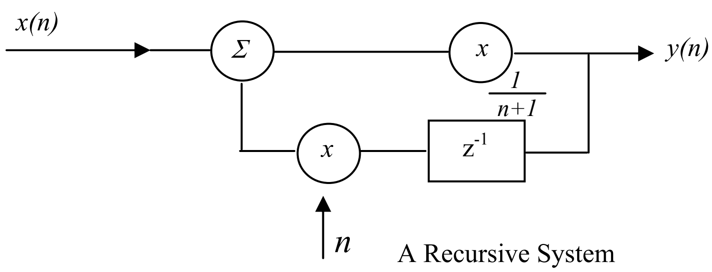
One approach is to use convolution sum but if the system is IIR, its practical implementation is impossible because it needs an infinite sum. Then to realize an IIR system, the solution is to use different equations.

There is a sub-class of recursive and non-recursive systems. Consider a commulative average:

$$y(n) = \frac{1}{n+1}\sum_{k=0}^{n} x(k) \quad 0 \le k \le n$$

$$(n+1)y(n) = \sum_{k=0}^{n} x(k) = \sum_{k=0}^{n-1} x(k) + x(n) = ny(n-1) + x(n)$$

$$\Rightarrow y(n) = \frac{n}{n+1} y(n-1) + \frac{1}{n+1} x(n)$$

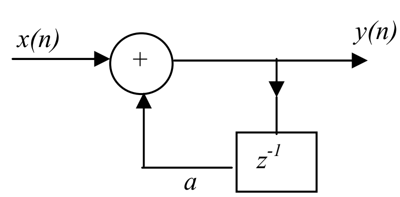


A Recursive System

Is this system LTI? No, as it is time-variant because of multiplying by *n*.

2.4.2 <u>LTI Systems Characterized by a Constant Coefficient Difference Equation</u>

Consider $y(n) = a\, y(n-1) + x(n)$



$$y(o) = ay(0) + x(1) = a^2 y(-1) + ax(0) + x(1)$$

$$\downarrow$$

$$y(n) = a^{n+1} y(-1) + a^n x(0) + a^{n-1} x(1) + \ldots$$

or $$y(n) = \underbrace{a^{n+1} y(-1)}_{zero-input\ response} + \underbrace{\sum_{K=0}^{n} a^K x(n-K)}_{zero-state\ or\ forced\ response} = y_{zi}(n) + y_{zs}(n) \quad **$$

The linearity applies to each of these responses separately. This system is linear and time-invariant.

Impulse Response – h(n) is simply equal to the zero-state response of the system. $y_{zs}(n)=\sum_{k=0}^{n}a^k x(n-k)$ Let $x(n)=\delta(n)$ then $\Rightarrow h(n)=\sum_{k=0}^{n}a^k\delta(n-k)=a^n \;\; n\ge 0$

Stability Example: is *y(n) = a y(n-1) + x(n)* stable?

Given a bounded input: $|x(n)|\le M_x<\infty$ for all $n\ge 0$, from (**) we have

$$y(n)\le\left|a^{n+1}y(-1)\right|+\left|\sum_{k=0}^{n}a^k x(n-k)\right|\le\left|a^{n+1}\right|\left|y(-1)\right|+M_x\frac{1-|a|^{n+1}}{1-|a|}=M_y$$

if n is finite, M_y is finite but if $n\to\infty$, M_y is bounded only if $|a|<1$.