Power Spectrum Estimation

<u>Parametric Methods</u> (Section 12.3)

Recall an AR model: $x(n) = -\sum_{k=1}^{P} a_p(k)x(n-1) + w(n)$ or $\sum_{k=o}^{P} a_p(k)x(n-k) = w(n)$,

where $a_p(0) = 1$. Or equivalently $A_p(z)\cdot X(z) = W(z)$ or $X(z) = \dfrac{W(z)}{A_p(z)} = H(z)\cdot W(z)$

$$w(n) \longrightarrow \boxed{\dfrac{1}{A_p(z)}} \longrightarrow x(n)$$

Remember that from Lecture 5, lecture notes, page 36, we have:

$$\Gamma_{xx}(\omega) = |H(\omega)|^2 \cdot \Gamma_{ww}(\omega) = \sigma_w^2 \cdot |H(\omega)|^2 \ (**)$$

$$\therefore \Gamma_{xx}(\omega) = PSD_x = \frac{\sigma_w^2}{|A_p(\omega)|^2} = \frac{\sigma_w^2}{\left|1 + \sum_{k=1}^{P} a_p(k)e^{-j\omega k}\right|^2}$$

This is for when *x(n)* is an AR process. In case that *x(n)* is not an AR process but we model it by an AR process, then it means $a_p(k)$ are the best approximate of the AR coefficients and σ_{wp}^2 is the MMSE. Then

$$P\hat{S}D_x = \hat{P}_x(\omega) = \frac{\sigma_{wp}^2}{\left|1 + \sum_{k=1}^{P} a_p(k)e^{-j\omega k}\right|^2}$$

Recall that $\sigma_{wp}^2 = MMSE = \hat{E}_P^f = \sigma_x^2 \prod_{k=1}^{P} \left|\left(1 - \underbrace{a_k(k)}_{k_m}\right)\right|^2$

** Proof of the equation (**)

$$\gamma_{xx}(l) = E\{x^*(n)x(n+l)\} = E\left\{\sum_{k=0}^{\infty} h^*(k)w^*(n-k)\cdot\sum_{m=0}^{\infty} h(m)w(n+l-m)\right\}$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} h^*(k)h(m)E\{w^*(n-k)w(n+l-m)\}$$

$$= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty} h^*(k)h(m)\gamma_{ww}(l-m+k)$$

A special case is when *w(n)* is white Gaussian noise. Then

$$\gamma_{ww}(l-m+k)=\begin{cases}\sigma_w^2 & \text{if } m=l+k\\ 0 & \text{else}\end{cases}$$

$$\therefore \gamma_{xx}(l)=\sum_{k=0}^{\infty}h^*(k)h(l+k)\sigma_w^2 = \sigma_w^2\cdot\gamma_{hh}(l)$$

Furthermore,

$$\Gamma_{xx}(\omega)=\sum_{l=-\infty}^{\infty}\gamma_{xx}(l)e^{-j\omega l} = \sum_{l}^{\infty}\left[\sum_{k}^{\infty}\sum_{m}^{\infty}h^*(k)h(m)\gamma_{ww}(l-m+k)\right]e^{-j\omega l}$$

$$=\sum_{k}^{\infty}\sum_{m}^{\infty}h^*(k)h(m)\sum_{l}^{\infty}\gamma_{ww}(l-m+k)e^{-j\omega l}$$

Let $l-m+k=u \Rightarrow l=u+m-k \Rightarrow e^{-j\omega l}=e^{-j\omega u}\cdot e^{-j\omega m}\cdot e^{jkl}$

$$=\sum_{k}^{\infty}h^*(k)e^{jkl}\sum_{m}^{\infty}h(m)e^{-jml}\sum_{u}^{\infty}\gamma_{ww}(u)e^{-j\omega u} = \underbrace{H(-\omega)}_{H^*(\omega)}\cdot H(\omega)\cdot\Gamma_{ww}(\omega)$$

$$\therefore \Gamma_{xx}(\omega)=\left|H(\omega)\right|^2\cdot\Gamma_{ww}(\omega)=\sigma_w^2\cdot\left|H(\omega)\right|^2$$

Non-Parametric Methods for PSD Estimation

Let's consider an observation of a stochastic process *x(n)*. Any observation is a finite record of the real process. Therefore, we can say:

$$\hat{x}(n)\equiv x_N(n)=\begin{cases}x(n) & n=0,\ldots..N-1\\ 0 & n\geq N\ n<0\end{cases}\equiv x(n)\cdot w(n)$$

where $w(n)=\begin{cases}1 & n=0,\ldots.N-1\\ 0 & else\end{cases}$ an ideal rectangle window.

Now let's see what is the effect of this limitation or truncation of the signal *x(n)*, on its PSD.

$$\tilde{X}(\omega)=\sum_{n=0}^{N-1}\tilde{x}(n)e^{-j\omega n}$$

By definition $PSD_{\tilde{x}}\equiv \tilde{P}_{xx}(\omega)=\frac{1}{N}\left|\tilde{x}(\omega)\right|^2\equiv\left|FFT(\hat{\gamma}_{xx}(l))\right|$. Therefore,

$$\left|\tilde{X}(\omega)\right|^2=\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}x(n).e^{-j\omega n}x^*(k)e^{j\omega k}$$

$$=\sum_{n=0}^{N-1}\sum_{k=0}^{N-1}x(n)x^*(k)e^{-j\omega(n-k)}$$

$$E\left\{\left|\widetilde{X}(\omega)\right|^2\right\} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} E\left\{x(n)x^*(k)\right\}e^{-j\omega(n-k)}$$
$$= \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} \gamma_{xx}(n-k)e^{-j\omega(n-k)}$$

Let $\ell = n-k \rightarrow k = n-\ell$

Range of n: $[0, N-1]$

$\rightarrow$ range of $\ell : \left[-N+1 \quad N-1\right]$

Range of k: $[0, N-1]$

$$\Rightarrow E\left\{\left|\widetilde{X}(\omega)\right|^2\right\} = \sum_{l=-N+1}^{0}\sum_{n=0}^{l+N-1} \gamma_{xx}(\ell)e^{-j\omega l} + \sum_{l=0}^{N-1}\sum_{n=l}^{N-1} \gamma_{xx}(\ell)e^{-j\omega l}$$

$$= \sum_{l=-N+1}^{0}(l+N)\gamma_{xx}(\ell)e^{-j\omega l} + \sum_{l=0}^{N-1}(N-l)\gamma_{xx}(\ell)e^{-j\omega l} = \sum_{l=-N+1}^{0}\left(N-|l|\right)\gamma_{xx}(\ell)e^{-j\omega l} + \sum_{l=0}^{N-1}\left(N-|l|\right)\gamma_{xx}(\ell)e^{-j\omega l}$$

$$= \sum_{l=-N+1}^{N-1}\left(N-|l|\right)\gamma_{xx}(\ell)e^{-j\omega l}$$

therefore, $E\left\{\tilde{P}_{xx}(\omega)\right\} = \dfrac{1}{N}E\left\{\left|\tilde{x}(\omega)\right|^2\right\} = \displaystyle\sum_{\ell=-N+1}^{N-1}\left(\underbrace{1-\frac{|\ell|}{N}}_{W_B(\ell)}\right)\gamma_{xx}(\ell)e^{-j\omega\ell}$

The above can also be reached by the other definition of power spectrum (Fourier Transform of the autocorrelation function):

$$PSD_{\tilde{x}} \equiv \tilde{P}_{xx}(\omega) \equiv \left|FFT\left(\hat{\gamma}_{xx}(l)\right)\right| = \sum_{l=-N+1}^{N-1}\hat{\gamma}_{xx}(l)e^{-j\omega l}$$

but $\tilde{\gamma}_{xx}(\ell) = \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-|\ell|-1}x^*(n)x(n+\ell)$

and therefore $\quad E\left\{\tilde{\gamma}_{xx}(\ell)\right\} = \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-|\ell|-1}E\left\{\underbrace{x^*(n)x(n+\ell)}_{\gamma_{xx}(\ell)}\right\} = \dfrac{N-|\ell|}{N}\gamma_{xx}(\ell)$

$$\Rightarrow E\left\{\tilde{P}_{xx}(\omega)\right\} = \sum_{l=-N+1}^{N-1}E\left\{\hat{\gamma}_{xx}(l)\right\}e^{-j\omega l} \quad \sum_{\ell=-N+1}^{N-1}\left(\underbrace{1-\frac{|\ell|}{N}}_{W_B(\ell)}\right)\gamma_{xx}(\ell)e^{-j\omega\ell}$$

Now $\displaystyle\lim_{N\to\infty} E\left\{\tilde{P}_{xx}(\omega)\right\} = \lim_{N\to\infty}\sum_{\ell=-N+1}^{N-1}\left(1-\frac{|\ell|}{N}\right)\gamma_{xx}(\ell)e^{-j\omega\ell}$

$$= \sum_{\ell=-\infty}^{+\infty} \gamma(\ell) e^{-j\omega\ell} = P_x(\omega) \quad \text{True PSD of } x(n)$$

Therefore, the effect of truncation on the power spectrum of the signal is that we get an asymptotically unbiased estimator of the true PSD of the signal. $P_x(\omega) \equiv \dfrac{\left|\tilde{X}(\omega)\right|^2}{N}$ is called **"Periodogram".**

As we said, it is an asymptotically unbiased estimator but its variance is not consistent or low because $\lim\limits_{N\to\infty} \text{var}\left\{\hat{P}_x(\omega)\right\} = P_x^2(\omega)$ and it doesn't go to zero. That's why Periodogram is referred as a noisy estimator of the true PSD of *x(n)*.

There are a few methods to remedy this variance problem.

Bartlet (Averaging Periodogram) Method

In order to reduce the variance of PSD estimators, Bartlet segmented the data to *K* segments each with length *M*. ($\Rightarrow K = \dfrac{N}{M}$, where *N* is the number of data samples), computed the Periodogram of each segment and then got the average of them as the PSD estimator.

data of each segment: $x_i(n) = x(n + iM)$ $\quad i = 0, 1, \ldots, K-1$ $\quad K$ = # of segments.

$n = 0, \ldots, M-1$ $\quad M$ = length of each segment

Periodogram of each segment:

$$P_x^{\,i}(\omega) = \frac{1}{M}\left|\sum_{n=0}^{M-1} x_i(n) e^{-j\omega n}\right|^2$$

Then the averaged Periodogram is: $P_x^{\,B}(\omega) = \dfrac{1}{K}\sum\limits_{i=0}^{K-1} p_x^{\,i}(\omega)$.

Lets see its statistical properties:

$E\left\{P_x^B(\omega)\right\} = \dfrac{1}{K}\sum\limits_{i=0}^{K-1} E\left\{p_x^{\,i}(\omega)\right\} = E\left\{p_x^{\,i}(\omega)\right\}$, which is the same as bias of Periodogram ($= \sum\limits_{\ell=-M+1}^{M-1} \left(\underbrace{1 - \dfrac{|\ell|}{M}}_{w_B(\ell)}\right) \gamma(\ell) e^{-j\omega\ell}$) and hence, asymptotically unbiased estimator .

The bias is $1-\frac{|\ell|}{M}$, which will go to 1 if $M \to \infty$

$W_B(\ell) = \begin{cases} 1-\frac{|\ell|}{M} & |\ell| \le M-1 \\ 0 & else \end{cases}$ is called Bartlet window, which is an inherent window as a result of truncation. Let's look at the variance of Bartlet Method:

$$\mathrm{var}\left\{P_x^B(\omega)\right\} = \frac{1}{K^2}\sum_{i=0}^{K-1}\mathrm{var}\left\{P_x^i(\omega)\right\} = \frac{1}{K}\mathrm{var}\left\{P_x^i(\omega)\right\}$$

Therefore, the variance is reduced by the factor *K*. So, it seems as we increase the number of segments (*K*), we reduce the variance of estimator more. But, this is at the cost of losing frequency resolution. What is frequency resolution? Does zero padding help that?